Parallel surfaces of cuspidal edges

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Introduction

Let $f : U \to \mathbf{R}^3$ be a smooth map from the domain $U \subset \mathbf{R}^2$. We call f a *front* if there exists a unit vector field ν along f such that $L := (f, \nu) : U \to T_1\mathbf{R}^3$ is a Legendrian immersion, where $T_1\mathbf{R}^3$ is the unit tangent bundle of $\mathbf{R}^3$ equipped with the canonical contact structure. For a front f, the function $\lambda : U \to \mathbf{R}$ defined as $\lambda(u,v) := \det(f_u, f_v, \nu)(u,v)$, where $f_u = \partial f/\partial u$, $f_v = \partial f/\partial v$ is called the *signed area density*. A point $p \in U$ is called a *singular point* of f if f is not an immersion at p. Let $S(f)$ be the set of singular points of f. A singular point $p \in S(f)$ is called *non-degenerate* if $d\lambda(p) \neq 0$ holds. If p is non-degenerate, then there exists a vector field η satisfying $df(\eta) = \mathbf{0}$ on $S(f)$ called the *null vector field*. A *cuspidal edge* is a map germ $\mathcal{A}$-equivalent to $(u,v) \mapsto (u, v^2, v^3)$ at $\mathbf{0}$, where two smooth map germs f and $g : (\mathbf{R}^2, \mathbf{0}) \to (\mathbf{R}^3, \mathbf{0})$ are *$\mathcal{A}$-equivalent* if there exist a diffeomorphism $S : (\mathbf{R}^2, \mathbf{0}) \to (\mathbf{R}^2, \mathbf{0})$ on the source and a diffeomorphism $T : (\mathbf{R}^3, \mathbf{0}) \to (\mathbf{R}^3, \mathbf{0})$ on the target such that $T \circ f = g \circ S$ holds.

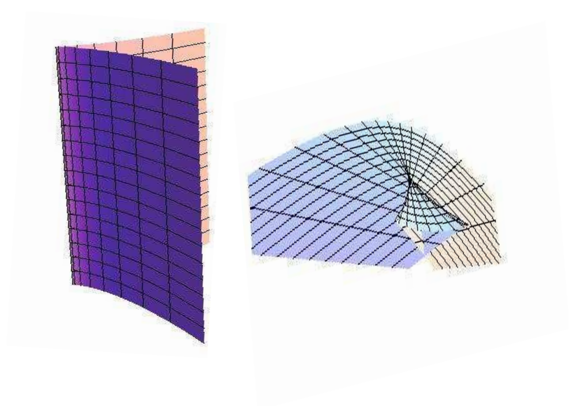

Figure 1: Cuspidal edge (left) and Swallowtail (right)

Theorem 1 ([KRSUY]). *Let the origin be a singular point of a front $f : U \to \mathbf{R}^3$.*
(1) f has a cuspidal edge at the origin if and only if $d\lambda(\eta) \neq 0$ at $\mathbf{0}$. In particular, at a cuspidal edge, the null direction and the singular direction are transversal.
(2) f has a swallowtail at the origin if and only if the conditions $d\lambda(\mathbf{0}) \neq 0$, $\eta\lambda(\mathbf{0}) = 0$ and $\eta\eta\lambda(\mathbf{0}) \neq 0$ are satisfied.

Cuspidal edges

Let $f = (f_1, f_2, f_3) : U \to \mathbf{R}^3$ be a cuspidal edge and $\nu = (\nu_1, \nu_2, \nu_3)$ a unit normal vector field of f. Then, by using only coordinate transformations on the source and isometries on the target, we obtain the following normal form for cuspidal edges (for details, see [MS]).

Proposition 1 ([MS]). *Let $f : (U; u, v) \to \mathbf{R}^3$ be a cuspidal edge. Using only coordinate transformations on the source and isometries on the target, f can be written as*

$$f(u,v) = \left(u, \frac{a_{20}}{2}u^2 + \frac{a_{30}}{6}u^3 + \frac{v^2}{2}, \frac{b_{20}}{2}u^2 + \frac{b_{30}}{6}u^3 + \frac{b_{12}}{2}uv^2 + \frac{b_{03}}{6}v^3\right) + h(u,v), \ (b_{20} \geq 0, b_{03} \neq 0) \quad (1)$$

where

$$h(u,v) = (0, u^4h_1(u), u^4h_2(u) + u^2v^2h_3(u) + uv^3h_4(u) + v^4h_5(u,v)),$$

with $h_i(u)$ $(1 \leq i \leq 4)$, $h_5(u,v)$ smooth functions.

We call this parametrization the *normal form* of cuspidal edges.
For later computations, we take a special coordinate system called *adapted coordinate system*.

Definition 1. A coordinate system $(U; u, v)$ is called *adapted* if it satisfies
1) the u-axis is the singular curve,
2) ∂_v gives a null vector field on the u-axis, and
3) there are no singular points other than the u-axis.

Using an adapted coordinate system, we can define the unit normal vector field along f as $\nu := f_u \times \tilde{f}_v / |f_u \times \tilde{f}_v|$, where $\tilde{f}_v = f_v/v$ and $\times$ means the vector product.

Remark 1. Let (u,v) be an adapted coordinate system on U. Since $\lambda_v \neq 0$, the pair $\{f_u, \tilde{f}_v, \nu\}$ is linearly independent and $f_{vv} = \tilde{f}_v$ holds on $\{v = 0\}$.

We define the coefficients of first and second fundamental forms forcuspidal edges as follows.

$$\begin{aligned} &\tilde{E} = \langle f_u, f_u \rangle, \ \tilde{F} = \langle f_u, \tilde{f}_v \rangle, \ \tilde{G} = \langle \tilde{f}_v, \tilde{f}_v \rangle, \\ &\tilde{L} = -\langle f_u, \nu_u \rangle, \ \tilde{M} = -\langle \tilde{f}_v, \nu_u \rangle, \ \tilde{N} = -\langle \tilde{f}_v, \nu_v \rangle, \end{aligned} \quad (2)$$

where $\langle , \rangle$ denotes the inner product in $\mathbf{R}^3$. Using (2), we have the following lemma.

Lemma 1. *The differentials ν_u and ν_v can be written as*

$$\nu_u = \frac{\tilde{F}\tilde{M} - \tilde{G}\tilde{L}}{\tilde{E}\tilde{G} - \tilde{F}^2} f_u + \frac{\tilde{F}\tilde{L} - \tilde{E}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2} \tilde{f}_v, \quad (3)$$

$$\nu_v = \frac{\tilde{F}\tilde{N} - v\tilde{G}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2} f_u + \frac{v\tilde{F}\tilde{M} - \tilde{E}\tilde{N}}{\tilde{E}\tilde{G} - \tilde{F}^2} \tilde{f}_v. \quad (4)$$

This lemma corresponds to the *Weingarten formula* for regular surfaces. Next we define the principal curvature for cuspidal edge. Let $f : U \to \mathbf{R}^3$ be a cuspidal edge, and let $(U; u, v)$ be an adapted coordinate system. Under these conditions, we define two functions

$$\tilde{\kappa}_1 := \frac{\tilde{A} + \tilde{B}}{2v(\tilde{E}\tilde{G} - \tilde{F}^2)}, \quad (5)$$

$$\tilde{\kappa}_2 := \frac{\tilde{A} - \tilde{B}}{2v(\tilde{E}\tilde{G} - \tilde{F}^2)} \quad (6)$$

on U,where

$$\begin{aligned} \tilde{A} &= \tilde{E}\tilde{N} - 2v\tilde{F}\tilde{M} + v\tilde{G}\tilde{L}, \\ \tilde{B} &= \sqrt{(\tilde{E}\tilde{N} - 2v\tilde{F}\tilde{M} + v\tilde{G}\tilde{L})^2 - 4v(\tilde{E}\tilde{G} - \tilde{F}^2)(\tilde{L}\tilde{N} - v\tilde{M}^2)}. \end{aligned}$$

In this case, we have the following.

Proposition 2. *Under the above conditions, if $\tilde{N}$ is positive (resp. negative), then $\tilde{\kappa}_2$ as in (6) converges i.e., it can be extended as a function near a cuspidal edge (resp. diverges) and $\tilde{\kappa}_1$ as in (5) diverges (resp. converges) on $S(f)$.*

By the construction, if $\tilde{N}$ is positive (resp. negative), $\tilde{\kappa}_2$ (resp. $\tilde{\kappa}_1$) can be regarded as the *principal curvature* of the cuspidal edge. Let us assume that $\tilde{N}$ is positive, that is, we consider $\tilde{\kappa}_2$ in the following. And we have the principal direction $\tilde{\boldsymbol{v}}$ with respect to $\tilde{\kappa}_2$ as follows: $\tilde{\boldsymbol{v}} = (\tilde{N} - \tilde{\kappa}_2 v\tilde{G}, -\tilde{M} + \tilde{\kappa}_2\tilde{F})$. Using the principal curvature $\tilde{\kappa}_2$ and the principal direction $\tilde{\boldsymbol{v}}$, we define the notion of *ridge points* for cuspidal edges.

Definition 2. The point $f(p)$ is called a *ridge point* for f relative to $\tilde{\boldsymbol{v}}$ if $\tilde{\boldsymbol{v}}\tilde{\kappa}_2(p) = 0$, where $\tilde{\boldsymbol{v}}\tilde{\kappa}_2$ is the directional derivative of $\tilde{\kappa}_2$ in $\tilde{\boldsymbol{v}}$. Moreover, $f(p)$ is called a *k-th order ridge point* for f relative to $\tilde{\boldsymbol{v}}$ if $\tilde{\boldsymbol{v}}^{(m)}\tilde{\kappa}_2(p) = 0$ $(1 \leq m \leq k)$ and $\tilde{\boldsymbol{v}}^{(k+1)}\tilde{\kappa}_2(p) \neq 0$, where $\tilde{\boldsymbol{v}}^{(m)}\tilde{\kappa}_2$ is the directional derivative of $\tilde{\kappa}_2$ with respect to $\tilde{\boldsymbol{v}}$ applied m times .

Using this notation, we have the following lemma:

Lemma 2. *Let $f : U \to \mathbf{R}^3$ be the normal form (1) of a cuspidal edge, $\tilde{\kappa}_2$ the principal curvature and $\tilde{\boldsymbol{v}}$ the principal direction corresponding to $\tilde{\kappa}_2$. Then the origin is a first order ridge point if and only if the following conditions hold:*

$$4b_{12}^3 + b_{30}b_{03}^2 = 0 \quad (7)$$

and

$$\begin{aligned} &b_{20}^3 b_{03}^4 + b_{20}(4b_{12}^2 + a_{20}b_{03}^2)^2 \\ &-8(b_{03}^4 h_2(0) + 4b_{12}^2 b_{03}^2 h_3(0) - 8b_{12}^3 b_{03} h_4(0) + 16b_{12}^4 h_5(0,0)) \neq 0. \end{aligned} \quad (8)$$

Parallel surfaces of cuspidal edges

We consider parallel surfaces of cuspidal edges. Let $f : U \to \mathbf{R}^3$ be cuspidal edges and ν be a unit normal vector of f. Then parallel surface $f_t : U \to \mathbf{R}^3$ of f is given by

$$f_t = f + t\nu,$$

where $t \in \mathbf{R}$ is a constant. Here we can take ν as the unit normal of f_t. The signed area density for f_t is

$$\lambda_t(u,v) = \det((f_t)_u, (f_t)_v, \nu)(u,v). \quad (9)$$

Since $S(f_t) = \{\lambda_t = 0\}$, to consider the singularity at $p \in U$, it is sufficient to take t which satisfies $\lambda_t(p) = 0$. If f is a normal form (1) of a cuspidal edge, then $\lambda_t(\mathbf{0}) = -b_{03}t(1 - b_{20}t)/2$. Hence we have $t = 0, 1/b_{20}$. We assume $b_{20} = \tilde{\kappa}_2(\mathbf{0}) \neq 0$. The case of $t = 0$ is the initial surface, so we set $t = t_0 = 1/\tilde{\kappa}_2(\mathbf{0}) (= 1/b_{20})$ and consider the condition that f_{t_0} at the origin is a swallowtail.

To apply Theorem 1, we consider the null vector field. We set the vector field $\eta_{t_0} := \ell_1(u,v)\partial_u + \ell_2(u,v)\partial_v$, where $\ell_i(u,v)$ $(i = 1, 2)$ are smooth functions. By direct computation, η_{t_0} can be written as

$$\eta_{t_0} = -\left(v + t_0\frac{v\tilde{F}\tilde{M} - \tilde{E}\tilde{N}}{\tilde{E}\tilde{G} - \tilde{F}^2}\right)\partial_u + t_0\frac{\tilde{F}\tilde{L} - \tilde{E}\tilde{M}}{\tilde{E}\tilde{G} - \tilde{F}^2}\partial_v. \quad (10)$$

By (10) and Theorem 1, we obtain the following:

Lemma 3. *Let $f : U \to \mathbf{R}^3$ be a cuspidal edge as given in (1) and f_{t_0} the parallel surface of f. Then f_{t_0} is a swallowtail at the origin if and only if the coefficients of the normal form satisfy*

$$b_{30} - a_{20}b_{12} \neq 0 \ \text{or} \ 4b_{12}^2 + a_{20}b_{03}^2 \neq 0, \quad (11)$$

$$4b_{12}^3 + b_{30}b_{03}^2 = 0, \quad (12)$$

and

$$\begin{aligned} &b_{20}^3 b_{03}^4 + b_{20}(4b_{12}^2 + a_{20}b_{03}^2)^2 \\ &-8(b_{03}^4 h_2(0) + 4b_{12}^2 b_{03}^2 h_3(0) - 8b_{12}^3 b_{03} h_4(0) + 16b_{12}^4 h_5(0,0)) \neq 0. \end{aligned} \quad (13)$$

Combining Lemmas 2 and 3, we have the relationship between the geometric properties of cuspidal edges and the singularities which appear in parallel surfaces.

Theorem 2. *Let $f : U \to \mathbf{R}^3$ be a cuspidal edge. Then the parallel surface f_{t_0} of f, where $t_0 = 1/\tilde{\kappa}_2(\mathbf{0})$, has a swallowtail at the origin if and only if the origin is a non-degenerate singular point of f_{t_0} and a first order ridge point of the initial surface f.*

Example 1. Let $f : U \to \mathbf{R}^3$ be a cuspidal edge given as $f(u,v) = (u, u^2/2 + u^3/3 + v^2/2, u^2 + v^3/3)$. The coefficients of f satisfy the conditions of Lemmas 2 and 3. The unit normal vector of f is

$$\nu = \frac{1}{\delta}(-2u + uv + u^2v, -v, 1),$$

where $\delta = \sqrt{1 + v^2 + (-2u + uv + u^2v)^2}$. Since $b_{20} = 2$, we take the parallel surface f_{t_0} as $f_{t_0} = f + \nu/2$. We can see a swallowtail singularity.

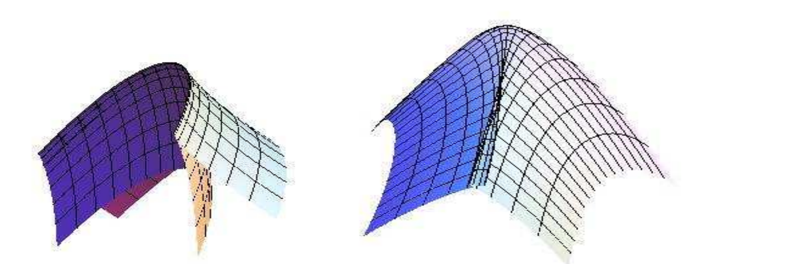

Figure 2: Initial cuspidal edge (left) and parallel surface (right).

Example 2. Let $f : U \to \mathbf{R}^3$ be a cuspidal edge defined by $f(u,v) := (u, u^2/2 + u^3/3 + v^2/2, u^2/2 + 4u^3/3 - uv^2 + v^3/3)$. In this case, $b_{20} = 1$ and the coefficients of f also satisfy the conditions in Lemmas 2 and 3. The unit normal vector of f is

$$\nu = \frac{1}{\delta}(-u(1 + 2u(3 + u)) + u(1 + u)v + v^2, 2u - v, 1),$$

where $\delta = \sqrt{1 + (-2u + v)^2 + (-u(1 + 2u(3 + u)) + u(1 + u)v + v^2)^2}$. The parallel surface f_{t_0} of f is given as $f_{t_0} = f + \nu$.

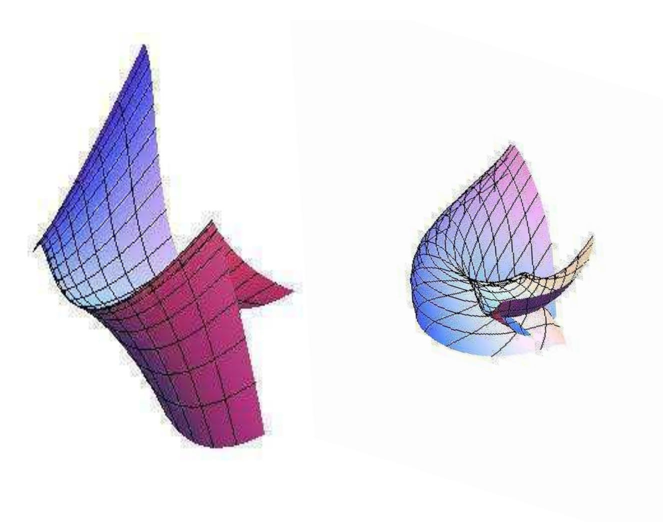

Figure 3: The left-hand side is f and the right-hand side is f_{t_0}.

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