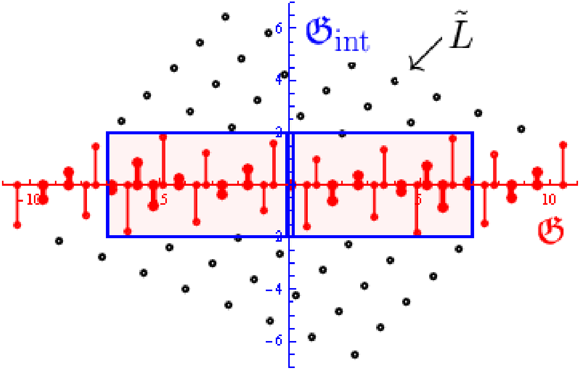
Randomized Model Set
切断射影集合, そのランダム化, 回折など

Yohji Akama[1] Shinji Iizuka[2]

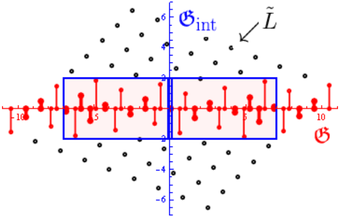
[1]Mathematical Institute, Tohoku University [2]Hitachi East Japan Solutions, Ltd.

October 25, 2009

モデル集合とその大域的秩序



切断射影スキーム (cut-and-project scheme (c.-p. scheme))



$$
\begin{array}{ccccc}
\mathfrak{G} & \xleftarrow[\text{射影}]{\Pi} & \mathfrak{G} \times \mathfrak{G}_{\mathrm{int}} & \xrightarrow[\text{射影}]{\Pi_{\mathrm{int}}} & \mathfrak{G}_{\mathrm{int}} \\
\cup & & \cup \text{ 格子} & & \cup \text{ 稠密} \\
L & \xleftarrow[\text{1-1}]{} & \tilde{L} & \longrightarrow & L^{\star} \\
\Cup & & \Cup & & \Cup \\
s & \longleftarrow\!\!\shortmid & (s, s^{\star}) & \longmapsto & s^{\star}
\end{array}
$$

$L^{\star}$ の稠密性 (i.e. $Cl(L^{\star}) = \mathfrak{G}_{\mathrm{int}}$) も要求していることに注意.

窓とは閉包を取るとコンパクトになる空でない集合.

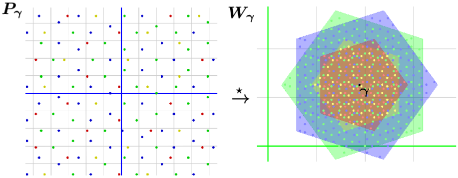
切断射影集合 (モデル集合) とは $\Lambda(W) := \{s \in L : s^{\star} \in W\}$

モデル集合の例 (Penrose tiling の頂点集合とその窓)

Penrose タイリングの頂点集合 $P_\gamma = \Lambda(\widetilde{L}, W_\gamma)$

[de Bruijn, '81]

$\widetilde{L}$ は $\zeta_5 = e^{2\pi i/5}$ とその Galois 共役から代数的につくられる.



$\mathfrak{G} = \mathbb{C}$　　　$\mathfrak{G}_{\mathrm{int}} = \mathbb{C} \times (\mathbb{Z}/5\mathbb{Z})$

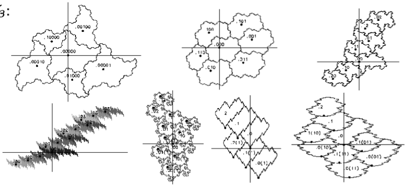
窓が 5 回対称性を持つため, 5 回対称性を持つ.

モデル集合 の例 (Pisot tiling の制御点集合 $\Lambda([0,1))$.)

Pisot フラクタルタイリング $\mathcal{T}_\beta$ の非対称性

任意の $f \in \mathrm{Aff}(\mathfrak{G})$ に対し $f(\mathcal{T}_\beta) = \mathcal{T}_\beta \Leftrightarrow f = \mathrm{id}$

$\mathcal{T}_\beta$:



高次元の Pisot タイリングでも成立

窓 $[0,1)$ を固定するアフィン変換なし. 故に, 制御点全体そして, タイリングを固定するアフィン変換なし [A.-Iizuka-Akazawa'08]

モデル集合の大域的基本的秩序

$\Lambda := \Lambda(W)$.

- Uniformly discrete:

$$\exists r > 0. \forall x \in \Lambda.(B(x, r) \cap \Lambda = \{x\}).$$

- Relatively dense(r.d.):

$$\exists R > 0. \forall x. B(x, R) \cap \Lambda \neq \emptyset.$$

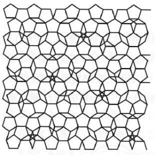
もし窓 W が内点を持てば relativley dense である.

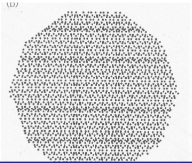
モデル集合における一様分布

モデル集合に対する Weyl の定理.
W の境界の測度は 0, かつ $f \in C(Cl(W), \mathbb{C})$ ならば

$$\lim_n \frac{\sum_{s \in \Lambda(W) \cap B(0,n)} f(s^\star)}{B(0,n) \text{ の体積}} = \frac{\int_W f(y)dy}{\tilde{L} \text{ の基本領域の体積}}.$$

Figure: Pentagonal 格子の頂点 $\Lambda(W)$ (左図) をスター写像で内空間 $\mathfrak{G}_{\text{int}}$ へ移したもの (右図).





回折測度

- $\Lambda(W)$ 上の原子の分布

$$\omega(x) = \sum_{s\in\Lambda(W)} w_s\delta_s(x), \quad w_s \text{ は理にかなった正の数列.}$$

$\int_A \varphi(x)\delta_s(x)dx = \varphi(s) \quad (s \in A).$

- 自己相関測度

$$\gamma_\omega(x) = \sum_{g\in\Lambda(W)-\Lambda(W)} \eta_\omega(g)\delta_g(x), \quad \sum_{s\in\Lambda(W)\cap B(0,n)} \frac{w_s w_{s+g}}{B(0,n)} \to \eta_\omega(g).$$

- **回折測度**(=↑のフーリエ変換)

$$\widehat{\gamma_\omega}(k) = \int e^{-2\pi\sqrt{-1}k\cdot x}\gamma_\omega(x)dx$$

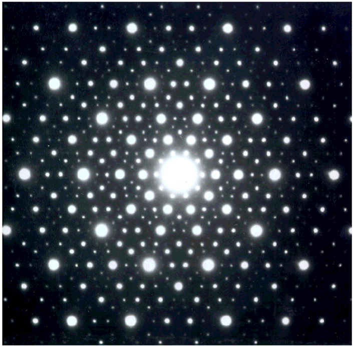
モデル集合の回折測度は純点的

一般に測度 $\theta(k)$ は $\sum_i a_i\delta_{k_i}(k) + h(k) + \sigma(k)$ ただし

$$\int \psi(k)\theta(k)dk = \sum_i a_i\psi(k_i) + \int \psi(k)h(k)dk. \quad (h : \text{可測})$$

とかける (純点成分, 絶対連続成分, 特異連続成分)
しかしモデル集合の回折測度には純点成分しかない
[Baake-Moody'04]

Randomized Model Set



A diffraction pattern of a quasicrystal

Motivation

- there is a quasicrystal such that the set of atoms' positions is not perfectly a model set. More specifically, every model set necessarily has a pure-point diffraction measure, but some real quasicrystal has a diffraction measure with not only Bragg peaks (pure-point component) but also diffuse scattering (absolutely continuous component). Physicists often associate the phoenomena to a probabilistic effect.
- mathematicians introduce, for their own purpose, a measure into the window of model sets.

Definition

A *randomized model set (r.m.s.)* is a 0, 1-valued stochastic process $\{X_s\}_{s\in\Lambda(W)}$ with p such that

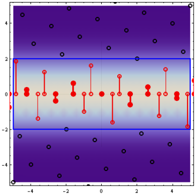
- $\Lambda(W)$ is a molde set ("*basis model set*");
- $p_g : Cl(W) \to [0,1]$ is continuous for each $g \in \Lambda(W) - \Lambda(W)$ and $\mathrm{E}[X_s X_{s-g}] = p_g(s^\star)$

$$\text{Intention :} \quad \begin{aligned} X_s = 1 &\iff s \text{ indeed appears.} \\ X_s = 0 &\iff s \text{ is defected.} \end{aligned}$$

R.m.s. obtained from $\Lambda(W)$ by random shift of W

$\{X_s \,;\, s \in \Lambda(W')\}$ such that

- $h(y)$ $(y \in R)$: probability density functions of shift y of W;
- $W' := W + R$;
- $$p_0(y) = 1_W * h,$$ $$p_g(y) = p_0(y)p_0(y - g^\star)$$ X_s is independent from X_{s-g}.



Localized Dependency in R.M.S.

Kolmogorov's strong law of large numbers(review)

$\forall$ *independent* $\{X_i\}_i$ *with* $\sum_{i=1}^{\infty} \frac{V[X_i]}{i^2} < \infty$,

$$\left| \frac{1}{n} \sum_{i=1}^{n} X_i - \frac{1}{n} \sum_{i=1}^{n} E[X_i] \right| \to 0, \qquad \textit{almost surely.}$$

A set $\{X_i\}_{i \in \mathbb{N}}$ of random variable is *independent*, iff for any finite subset $\{X, X', X'', \ldots\}$,

$$\begin{aligned} & P(X \in B \ \& \ X' \in B' \ \& \ X'' \in B'' \ \& \ \ldots) \\ =& P(X \in B) P(X' \in B') P(X'' \in B'') \cdots, \end{aligned}$$

where $B, B', B'', \ldots$ are Borel sets.

Localized dependency condition

There is a finite set D such that each site s has probabilistic influence on, at most, sites belonging to $s + D$. More precisely,

$$0 \in \exists \textit{ finite } D = -D \subseteq \Lambda(W) - \Lambda(W),$$
$$\forall \textit{ finite } P, Q \subseteq \Lambda(W) \textit{ with } (P - Q) \cap D = \emptyset$$
$$\{(X_s)_{s \in P},\ (X_s)_{s \in Q}\} \textit{ is independent.}$$

D=$\{0\}$ $\iff$ r.m.s. $\{X_s\ ;\ s \in \Lambda(W)\}$ is independent.

Nice r.m.s. : R.m.s. with localized dependency

A r.m.s. over a c.-p. scheme $(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \tilde{L})$ is *nice*, if $\exists D$ s.t.

- D satisfies localized dependency condition; and
- either $D = \{0\}$ or $\tilde{L}$ is finitely generated.

Let us call D a *dependency set* of the nice r.m.s.

$$\begin{array}{ccccc}
\mathfrak{G} & \xleftarrow[\text{proj.}]{\Pi} & \mathfrak{G} \times \mathfrak{G}_{\mathrm{int}} & \xrightarrow[\text{proj.}]{\Pi_{\mathrm{int}}} & \mathfrak{G}_{\mathrm{int}} \\
\cup & & \cup \ \text{lattice} & & \cup \ \text{dense} \\
L & \xleftarrow[\text{1-1}]{} & \tilde{L} & \longrightarrow & L^\star \\
\rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\
s & \longmapsfrom & (s, s^\star) & \longmapsto & s^\star
\end{array}$$

Let μ, θ be Haar measures of $\mathfrak{G}$ and $\mathfrak{G}_{\mathrm{int}}$, respectively.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

$\forall$ **nice r.m.s.** $\{X_s \,;\, s \in \Lambda(W)\}$ **over a c.-p.s.** $(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \tilde{L})$
$\exists$ c.-p. s. $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ with $\mathfrak{H}_{\mathrm{int}} \subset \mathfrak{G}_{\mathrm{int}}$, $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
   1. S_C is a translation of an r.m.s. over $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ and
   2. $\{X_s \,;\, s \in S_C\}$ is independent;
2. if $\theta(\partial W) = 0$, then every S_C has density $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$; and
3. if W has an inner point, some S_C is relatively dense.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

$\forall$ nice r.m.s. $\{X_s \; ; \; s \in \Lambda(W)\}$ over a c.-p.s. $(\mathfrak{G}, \mathfrak{G}_{\text{int}}, \tilde{L})$

$\exists$ **c.-p. s.** $(\mathfrak{G}, \mathfrak{H}_{\text{int}}, \tilde{M})$ **with** $\mathfrak{H}_{\text{int}} \subset \mathfrak{G}_{\text{int}}$**,** $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
   1. S_C is a translation of an r.m.s. over $(\mathfrak{G}, \mathfrak{H}_{\text{int}}, \tilde{M})$ and
   2. $\{X_s \; ; \; s \in S_C\}$ is independent;
2. if $\theta(\partial W) = 0$, then every S_C has density $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$; and
3. if W has an inner point, some S_C is relatively dense.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

- $\forall$ nice r.m.s. $\{X_s \,;\, s \in \Lambda(W)\}$ over a c.-p.s. $(\mathfrak{G}, \mathfrak{G}_{\rm int}, \tilde{L})$
- $\exists$ c.-p. s. $(\mathfrak{G}, \mathfrak{H}_{\rm int}, \tilde{M})$ with $\mathfrak{H}_{\rm int} \subset \mathfrak{G}_{\rm int}$, $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
   1. S_C **is a translation of an r.m.s. over** $(\mathfrak{G}, \mathfrak{H}_{\rm int}, \tilde{M})$ and
   2. $\{X_s \,;\, s \in S_C\}$ is independent;
2. if $\theta(\partial W) = 0$, then every S_C has density $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$; and
3. if W has an inner point, some S_C is relatively dense.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

$\forall$ nice r.m.s. $\{X_s \ ; \ s \in \Lambda(W)\}$ over a c.-p.s. $(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \tilde{L})$

$\exists$ c.-p. s. $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ with $\mathfrak{H}_{\mathrm{int}} \subset \mathfrak{G}_{\mathrm{int}}$, $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
   1. S_C is a translation of an r.m.s. over $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ and
   2. $\{X_s \ ; \ s \in S_C\}$ **is independent;**
2. if $\theta(\partial W) = 0$, then every S_C has density $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$; and
3. if W has an inner point, some S_C is relatively dense.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

$\forall$ nice r.m.s. $\{X_s \,;\, s \in \Lambda(W)\}$ over a c.-p.s. $(\mathfrak{G}, \mathfrak{G}_{\rm int}, \tilde{L})$
$\exists$ c.-p. s. $(\mathfrak{G}, \mathfrak{H}_{\rm int}, \tilde{M})$ with $\mathfrak{H}_{\rm int} \subset \mathfrak{G}_{\rm int}$, $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
   1. S_C is a translation of an r.m.s. over $(\mathfrak{G}, \mathfrak{H}_{\rm int}, \tilde{M})$ and
   2. $\{X_s \,;\, s \in S_C\}$ is independent;
2. **if** $\theta(\partial W) = 0$**, then every** S_C **has density** $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$ **;** and
3. if W has an inner point, some S_C is relatively dense.

Decomposition into translated i.r.m.s.'s, each having density

Decomposition Theorem.

$\forall$ nice r.m.s. $\{X_s\ ;\ s \in \Lambda(W)\}$ over a c.-p.s. $(\mathfrak{G}, \mathfrak{G}_{\mathrm{int}}, \tilde{L})$
$\exists$ c.-p. s. $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ with $\mathfrak{H}_{\mathrm{int}} \subset \mathfrak{G}_{\mathrm{int}}$, $\tilde{M} \subset \tilde{L}$;

such that

1. $\Lambda(W)$ is a disjoint union of S_C over $C \in \tilde{L}/\tilde{M}$, such that for each C
    1. S_C is a translation of an r.m.s. over $(\mathfrak{G}, \mathfrak{H}_{\mathrm{int}}, \tilde{M})$ and
    2. $\{X_s\ ;\ s \in S_C\}$ is independent;
2. if $\theta(\partial W) = 0$, then every S_C has density $\lim_n \frac{\#(S_C \cap D_n)}{\mu(D_n)}$; and
3. **if W has an inner point, some S_C is relatively dense.**

Global Order of Nice Randomized Model Set

How global order of model sets survive
against probabilistic disturbance?

Relatively dense with negligible error

0, 1-valued stochastic process $\{X_s \; ; \; s \in \Lambda\}$ with $\Lambda \subset \mathfrak{G}$ is said to be *relatively dense with negligible error*, if

$$\forall \varepsilon > 0 \; \exists \text{cpt } K \subset \mathfrak{G} \; \forall x \in \mathfrak{G}. \; P\left(\exists s \in (x + K) \cap \Lambda. \, X_s = 1\right) \geq 1 - \varepsilon.$$

Theorem. A nice r.m.s. is relatively dense with negligible error, if

- it is independent, the basis model set $\Lambda(W)$ is relatively dense and $\inf_{y \in Cl(W)} p_0(y) > 0$; or
- $\inf_{y \in U} p_0(y) > 0$ for $\exists$ nonempty open $U \subset W$.

By D.Th(4).

Distribution in r.m.s.

Weyl's Theorem for r.m.s.

For any nice r.m.s. $\{X_s ; s \in \Lambda(W)\}$, if W : measurable but $\theta(\partial W) = 0$, and $f \in C(Cl(W), \mathbb{C})$, then almost surely

$$\lim_n \frac{\sum_{s \in \Lambda(W) \cap D_n} X_s f(s^\star)}{\mu(D_n)} = \frac{\int_W f(y) p_0(y) d\theta(y)}{|\tilde{L}|_{\mu \otimes \theta}},$$

where $(D_n)_n$ is a van Hove sequence, and

$|\tilde{L}|_{\mu \otimes \theta} = (\mu \otimes \theta)$(the lattice $\tilde{L}$'s fundamental domain).
$p_0(s^*) = \mathrm{E}[X_s]$

Proof of Weyl's thm for r.m.s. $(f = 1)$.

If $S_C \neq \emptyset$, then

$$\left| \frac{\sum_{s \in S_C \cap D_n} X_s}{\#(S_C \cap D_n)} - \frac{\sum_{s \in S_C \cap D_n} \mathrm{E}[X_s]}{\#(S_C \cap D_n)} \right| \overset{n \to \infty}{\longrightarrow} 0 \quad \text{(a.s.) by Kolmo.'s..}$$

By D.Th(3)

$$\frac{\sum_{s \in S_C \cap D_n} |X_s - \mathrm{E}[X_s]|}{\mu(D_n)} = \frac{\sum_{s \in S_C \cap D_n} |X_s - \mathrm{E}[X_s]|}{\#(S_C \cap D_n)} \frac{\#(S_C \cap D_n)}{\mu(D_n)} \to 0$$

As the m.s. $\Lambda(W)$ is a disjoint union of S_C's, and has density

$$\left| \frac{\sum_{s \in \Lambda(W) \cap D_n} X_s}{\mu(D_n)} - \frac{\sum_{s \in \Lambda(W) \cap D_n} \mathrm{E}[X_s]}{\mu(D_n)} \right| \to 0$$

Since $E[X_s] = p_0(s^\star)$ for some continuous $p_0 : Cl(W) \to \mathbb{R}$, we have done by Weyl's thm for m.s.

Diffraction of nice r.m.s.

Theorem. Let $\{X_s\}_{s\in\Lambda(W)}$ be a nice r.m.s. such that $\Lambda(W)$ have no infinite arithmetical sequence. And let $f : \mathfrak{G}_{\mathrm{int}} \to \mathbb{R}_{\geq 0}$ be supported and conti. on $Cl(W)$. Then diffract. meas. $\widehat{\gamma_\omega}$ of Dirac comb $\omega = \sum_{s\in\Lambda(W)} X_s f(s^\star)\delta_s$ consists almost surely of

- *absolutely continuous component*, whose inverse Fourier transform is the Dirac comb over the smallest dependency set D where the coefficient at $g \in D$ is the spatial average of the covariance between $X_s f(s^\star)$ and $X_{s-g} f(s^\star - g^\star)$, over $s \in \Lambda(W)$.
- pure-point component, which is $\widehat{\gamma_{\mathrm{E}[\omega]}}$ where

$$\mathrm{E}[\omega] := \sum_{s\in\Lambda(W)} p_0(s^\star) f(s^\star)\delta_s.$$