量子群と q 差分量子 Weyl 群双有理作用

Quantum groups and quantized q-difference birational Weyl group actions

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Quantum birational Weyl group action

GCM $[a_{ij}]$	$d_i a_{ij} = d_j a_{ji}$, $\langle h_i, \alpha_i \rangle = a_{ij}$, $\alpha_i^\vee = d_i^{-1} \alpha_i$
Ore domain	q-Serre: $\sum_k (-1)^k \left[{1-a_{ij} \atop k} \right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \ (i \neq j)$
$\mathcal{A}_q = \langle e_i, a_i \rangle_{i \in I}$	Parameters $a_i = q^{\alpha_i}$: $a_i e_j = e_j a_i$
$\downarrow$	
Field of fractions	$\mathcal{K}_q = Q(\mathcal{A}_q) = \{\, ab^{-1} \mid a, b \in \mathcal{A}_q,\ b \neq 0 \,\}$
$\downarrow$	
$e_i^n e_j e_i^{-n}$	is an n-independent rational function of q_i^n
$\downarrow$	$\downarrow$ $\quad q_i^n \mapsto a_i = q_i^{\alpha_i^\vee} \qquad (q_i = q^{d_i})$
Weyl group action	$s_i(e_j) = e_i^{\alpha_i^\vee} e_j e_i^{\alpha_i^\vee}$
on $\mathcal{K}_q$	$s_i(a_j) = a_i^{-a_{ij}} a_j \ (\iff s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i)$

Both q-difference analogue and canonical quantization of the birational Weyl group action given by Noumi-Yamada math/0012028.

Explicit formulae for $s_i(e_j) = e_i^{\alpha_i^\vee} e_j e_i^{\alpha_i^\vee}$

- $[x,y]_q := xy - qyx, \quad q(k) := q_i^{2k+a_{ij}} \qquad (i \ne j)$
- Define $(\operatorname{ad}_q e_i)^k(e_j)$ for $k = 0, 1, 2, \dots$ by
  $(\operatorname{ad}_q e_i)^k(e_j) = [e_i, [\cdots, [e_i, [e_i, e_j]_{q(0)}]_{q(1)} \cdots]_{q(k-2)}]_{q(k-1)}$.
- Then $(\operatorname{ad}_q e_i)^k(e_j) = \displaystyle\sum_{\nu=0}^{k} (-1)^\nu q_i^{\nu(k-1+a_{ij})} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q_i} e_i^{k-\nu} e_j e_i^\nu$.
- q-Serre relations $\iff$ $(\operatorname{ad}_q e_i)^k(e_j) = 0$ if $i \ne j$ and $k > -a_{ij}$.
- $s_i(e_j) = \begin{cases} e_i & (i = j), \\ \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee - k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\operatorname{ad}_q e_i)^k(e_j) e_i^{-k} & (i \ne j). \end{cases}$

Quantum geometric crystal structure on $\mathcal{K}_q$

GCM $[a_{ij}]$	$d_i a_{ij} = d_j a_{ji}$, $\langle h_i, \alpha_i \rangle = a_{ij}$, $\alpha_i^\vee = d_i^{-1}\alpha_i$
Ore domain	q-Serre: $\sum_k (-1)^k \left[{1-a_{ij} \atop k}\right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0$ $(i \neq j)$
$\mathcal{A}_q = \langle e_i, a_i \rangle_{i \in I}$	Parameters $a_i = q^{\alpha_i}$: $a_i e_j = e_j a_i$
$\downarrow$	
Field of fractions	$\mathcal{K}_q = Q(\mathcal{A}_q) = \{\, ab^{-1} \mid a, b \in \mathcal{A}_q,\ b \neq 0 \,\}$
$\downarrow$	
$e_i^n e_j e_i^{-n}$	is an n-independent rational function of q_i^n
$\downarrow$	$\downarrow$ $\quad q_i^n \mapsto t$
Quantum geometric	$\mathbf{e}_i^t(e_j) = e_i^n e_j e_i^{-n}\vert_{q_i^n \mapsto t}$
crystal str. on $\mathcal{K}_q$	$\mathbf{e}_i^t(a_j) = t^{-a_{ij}} a_j$

The Verma relations for $\mathbf{e}_i^t \implies$ Weyl group action $s_i = \mathbf{e}_i^{a_i}$

Generalization

q-Serre relations	$\sum_k (-1)^k \left[\begin{smallmatrix} 1-a_{ij} \\ k \end{smallmatrix}\right]_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \ (i \neq j)$
$\downarrow$	
Verma relations	$e_i^k e_j^{k+l} e_i^l = e_j^l e_i^{k+l} e_j^k$ if $a_{ij}a_{ji} = 1$, etc.
$\downarrow$	
Assumption	$e_i^n x e_i^{-n}$ is a rational fucntion of q_i^n.
$\downarrow$	$\downarrow \quad q_i^n \mapsto t$
Quantum geomtric crystal	$\mathbf{e}_i^t(x) = e_i^n x e_i^{-n}\vert_{q_i^n \mapsto t}$ $\mathbf{e}_i^t(a_j) = t^{-a_{ij}} a_j$ (action on parameters)
$\downarrow$	
Weyl group action	$s_i = \mathbf{e}_i^{\alpha_i^\vee} \quad (a_i = q_i^{\alpha_i^\vee})$

Actions of the lattice parts of affine Weyl groups

$\to$ q-difference quantum Painlevé systems

Quantum Schubert cell

Reduced expression of $w \in W$: $w = s_{i_1} \cdots s_{i_N}$, $\mathbf{i} = (i_1, \dots, i_N)$.

$\downarrow$

$\mathcal{A}_\mathbf{i} = \langle x_\nu, a_i \rangle$, $\mathcal{K}_\mathbf{i} = Q(\mathcal{A}_q) = \{\, ab^{-1} \mid a, b \in \mathcal{A}_\mathbf{i},\ b \neq 0 \,\}$

Defining relations: $x_\nu x_\mu = q_{\mu\nu} x_\mu x_\nu$ $(\mu < \nu)$, $a_i \in$ center.

$q_{\mu\nu} := q^{b_{i_\mu i_\nu}}$, $b_{ij} := d_i a_{ij}$.

$\downarrow$

$(x_1, \dots, x_N) \mapsto e_{q_{i_1}}(x_1 F_{i_1}) \cdots e_{q_{i_N}}(x_N F_{i_N})$

is quantization of a positive structure of a Schubert cell.

$\downarrow$

$e_i := \sum_{i_\nu = i} x_\nu$ $\begin{cases} q\text{-Serre relations for } e_i, \\ e_i^n x_\nu e_i^{-n} \text{ is a rational function of } q_i^n. \end{cases}$

$\downarrow$

Quantum geometric crystal structure on $\mathcal{K}_\mathbf{i}$.

Explicit formulae for $\mathrm{e}_i^t(x_\nu)$

- $X := \sum_{i_\mu=i,\ \mu<\nu} x_\mu, \quad Y := \sum_{i_\mu=i,\ \mu>\nu} x_\mu.$.
- Then $e_i = X + \delta_{i_\nu i} x_\nu + Y$.
- If $i_\nu = i$, then

$$\mathbf{e}_i^t(x_\nu) = t^{-2} x_\nu \frac{1 + q_i^2(x_\nu + q_i^2 Y)X^{-1}}{1 + q_i^2 t^{-2}(x_\nu + q_i^2 Y)X^{-1}} \frac{1 + q_i^2 Y(X + x_\nu)^{-1}}{1 + q_i^2 t^{-2} Y(X + x_\nu)^{-1}},$$

- If $i_\nu \neq i$, then $-a_{ii_\nu} \geqq 0$ and

$$\mathbf{e}_i^t(x_\nu) = t^{-a_{ii_\nu}} x_\nu \prod_{k=1}^{-a_{ii_\nu}} \frac{1 + q_i^{-2(k-1)} t^{-2} Y X^{-1}}{1 + q_i^{-2(k-1)} Y X^{-1}}.$$

Remarks

- All the expressions for $\mathbf{e}_i^t(x_\nu)$ are **subtraction-free**.
  $\mathcal{K}_\mathbf{i}$ depends only on $w \in W$ up to canonical **positive** isomorphisms.
  $\downarrow$ generalization
  Various **quantum positive geometric crystals**
  $\downarrow$ classical limit
  Various positive geometric crystals
  $\downarrow$ ultra-discretization
  Various crystals

- φ_i and ε_i.
  - $\mathbf{e}_i^t(e_i) = e_i, \quad \mathbf{e}_i^t(q^{\alpha_i}) = t^{-2} q^{\alpha_i}.$
  - $\varepsilon_i :=$ const.$q^{\alpha_i} e_i, \quad \varphi_i :=$ const.$q^{-\alpha_i} e_i. \quad (\varepsilon_i = q^{2\alpha_i}\varphi_i)$
  - Then $\quad \mathbf{e}_i^t(\varphi_i) = t^2 \varphi_i, \quad \mathbf{e}_i^t(\varepsilon_i) = t^{-2}\varepsilon_i.$
  - quantum t^{-2}, $q^{2\alpha_i} \longleftrightarrow$ classical "c", "α_i"

Files

- Old version of this file $\longrightarrow$
http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100924_Nagoya.pdf

- Quantum M-matrix for A_∞ case $\longrightarrow$ §1.6 of
http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100630_Osaka.pdf

- Quantization of the birational action of $W(A^{(1)}_{m-1}) \times W(A^{(1)}_{n-1})$ given by Kajiwara-Noumi-Yamada nlin/0106029 for mutually prime m, n
$\to$ http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100630_WxW.pdf

- Theory of quantum geometric crystals $\longrightarrow$ in preparation

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For more details see the following pages.

Symmetrizable GCM and root datum

- Let $A = [a_{ij}]_{i,j\in I}$ be a symmetrizable GCM:
  - $a_{ii} = 2, \quad a_{ij} \leqq 0 \ (i \neq j), \quad a_{ij} = 0 \iff a_{ji} = 0;$
  - $d_i a_{ij} = d_j a_{ji}, \quad d_i \in \mathbb{Z}_{>0}.$
- Let $(\langle\,,\rangle : Q^\vee \times P \to \mathbb{Z},\ \{h_i\}_{i\in I} \subset Q^\vee,\ \{\alpha_i\}_{i\in I} \subset P)$ be a root datum:
- finitely generated free $\mathbb{Z}$-modules $Q^\vee$, P and perfect bilinear pairing $\langle\,,\rangle : Q^\vee \times P \to \mathbb{Z}$.
  - $\{h_i\}_{i\in I} \subset Q^\vee$ is called a set of simple coroots. $Q^\vee$ is called a coroot lattice.
  - $\{\alpha_i\}_{i\in I} \subset P$ is called a set of simple roots. P is called a weight lattice.
  - $\langle h_i, \alpha_j\rangle = a_{ij}$.

The group algebra $\mathbb{F}[q^P]$ of the weight lattice P

- Base field $\mathbb{F} := \mathbb{Q}(q)$.
- $\mathbb{F}[q^P] := \bigoplus_{\lambda \in P} \mathbb{F} q^\lambda, \quad q^\lambda q^\mu = q^{\lambda+\mu} \quad (\lambda, \mu \in P)$.
- $[x]_q := \dfrac{q^x - q^{-x}}{q - q^{-1}}, \quad [k]_q! := [1]_q [2]_q \cdots [k]_q \quad (k \in \mathbb{Z}_{\geqq 0})$.
- $\begin{bmatrix} x \\ k \end{bmatrix}_q := \dfrac{[x]_q [x-1]_q \cdots [x-k+1]_q}{[k]_q!}$ (q-binomial coefficients).
- $q_i := q^{d_i}, \quad \alpha_i^\vee := d_i^{-1} \alpha_i$ ($=$ a simple coroot).

Remark. $q^{\pm d_i \alpha_i^\vee} = q^{\pm \alpha_i} \in \mathbb{F}[q^P] \implies \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} \in \mathbb{F}[q^P]$.

Quantum algebra $\mathcal{A}_q = \langle\, q^\lambda, e_i \mid \lambda \in P, i \in I \,\rangle$

Assumptions.

(1) $\mathcal{A}_{q,0}$ is an associative algebra over $\mathbb{F}$ generated by $e_i \neq 0$ $(i \in I)$.

(2) q-Serre relations: $\displaystyle\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0 \quad (i \neq j)$.

(3) $\mathcal{A}_q := \mathbb{F}[q^P] \otimes_{\mathbb{F}} \mathcal{A}_{q,0}$ is an Ore domain.

Identification. $q^\lambda = q^\lambda \otimes 1 \in \mathcal{A}_q, \quad e_i = 1 \otimes e_i \in \mathcal{A}_q$.

Remark. $q^\lambda e_i = e_i q^\lambda$ in $\mathcal{A}_q$.

- $Q(\mathcal{A}_q) :=$ (the quotient skew field of $\mathcal{A}_q$) $= \{\, as^{-1} \mid a, s \in \mathcal{A}_q,\ s \neq 0 \,\}$.

Example. The root datum is of finite or affine type

$\Longrightarrow$ $\mathcal{A}_{q,0} = U_q(\mathfrak{n}_+)$ satisfies all the assumptions above.

Iterated adjoint by e_i

- Assume $i \neq j$.
- $[x, y]_q := xy - qyx, \quad q(k) := q_i^{2k+a_{ij}}$
- Define $(\mathrm{ad}_q\, e_i)^k(e_j)$ for $k = 0, 1, 2, \dots$ by

$$
\begin{aligned}
(\mathrm{ad}_q\, e_i)^0(e_j) &= e_j, \\
(\mathrm{ad}_q\, e_i)^1(e_j) &= [e_i, e_j]_{q(0)}, \\
(\mathrm{ad}_q\, e_i)^2(e_j) &= [e_i, [e_i, e_j]_{q(0)}]_{q(1)}, \dots, \\
(\mathrm{ad}_q\, e_i)^k(e_j) &= [e_i, [\cdots, [e_i, [e_i, e_j]_{q(0)}]_{q(1)} \cdots]_{q(k-2)}]_{q(k-1)}.
\end{aligned}
$$

- Then $(\mathrm{ad}_q\, e_i)^k(e_j) = \displaystyle\sum_{\nu=0}^{k} (-1)^\nu q_i^{\nu(k-1+a_{ij})} \begin{bmatrix} k \\ \nu \end{bmatrix}_{q_i} e_i^{k-\nu} e_j e_i^\nu$.
- q-Serre relations $\iff$ $(\mathrm{ad}_q\, e_i)^k(e_j) = 0$ if $i \neq j$ and $k > -a_{ij}$.

Conjugation by powers of e_i

- For $n = 0, 1, 2, \ldots$,

$$e_i^n e_j e_i^{-n} = \begin{cases} e_i & (i = j), \\ \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, e_i)^k (e_j) e_i^{-k} & (i \neq j). \end{cases}$$

- Define $e_i^{\alpha_i^\vee} e_j e_i^{-\alpha_i^\vee} \in Q(\mathcal{A}_q)$ by

$$e_i^{\alpha_i^\vee} e_j e_i^{-\alpha_i^\vee} = \begin{cases} e_i & (i = j), \\ \displaystyle\sum_{k=0}^{-a_{ij}} q_i^{(k+a_{ij})(\alpha_i^\vee-k)} \begin{bmatrix} \alpha_i^\vee \\ k \end{bmatrix}_{q_i} (\mathrm{ad}_q\, e_i)^k (e_j) e_i^{-k} & (i \neq j). \end{cases}$$

- $x \mapsto e_i^n x e_i^{-n}$ is an algebra automorphism of $Q(\mathcal{A}_q)$
  $\implies e_j \mapsto e_i^{\alpha_i^\vee} e_j e_i^{-\alpha_i^\vee}$ is uniquely extended to an alg. autom. of $Q(\mathcal{A}_q)$.

Qauntized birational Weyl group action

Theorem 1. The algebra automorphim s_i of $Q(\mathcal{A}_q)$ can be defined by

$$s_i(e_j) = e_i^{\alpha_i^\vee} e_j e_i^{-\alpha_i^\vee} \quad (i \in I), \qquad s_i(q^\lambda) = q^{\lambda - \langle h_i, \lambda \rangle \alpha_i} \quad (\lambda \in P).$$

Then $\{s_i\}_{i \in I}$ satisfies the defining relations of the Weyl group W:

$$s_i s_j = s_j s_i \ (a_{ij} a_{ji} = 0), \quad s_i s_j s_i = s_j s_i s_j \ (a_{ij} a_{ji} = 1),$$
$$s_i s_j s_i s_j = s_j s_i s_j s_i \quad (a_{ij} a_{ji} = 2),$$
$$s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i \quad (a_{ij} a_{ji} = 3), \quad s_i^2 = 1.$$

Thus we obtain the action of the Weyl group W on $Q(\mathcal{A}_q)$. □

Remark. This is a q-difference version of quantization of the birational Weyl group action given by Noumi-Yamada math/0012028.

The Verma relations of $\{e_i\}_{i\in I}$

- (Lusztig's book (1993)) q-Serre relations of $\{e_i\}_{i\in I}$ implies
  - $(a_{ij}, a_{ji}) = (0,0) \implies e_i^k e_j^l = e_j^l e_i^k$,
  - $(a_{ij}, a_{ji}) = (-1,-1) \implies e_i^k e_j^{k+l} e_i^l = e_j^l e_i^{k+l} e_j^k$,
  - $(a_{ij}, a_{ji}) = (-1,-2) \implies e_i^k e_j^{2k+l} e_i^{k+l} e_j^l = e_j^l e_i^{k+l} e_j^{2k+l} e_i^k$,
  - $(a_{ij}, a_{ji}) = (-1,-3)$
    $\implies e_i^k e_j^{3k+l} e_i^{2k+l} e_j^{3k+2l} e_i^{k+l} e_j^l = e_j^l e_i^{k+l} e_j^{3k+2l} e_i^{2k+l} e_j^{3k+l} e_i^k$.

These relations are called the Verma relations.

- The Verma relations
  $\implies$ $\{s_i\}_{i\in I}$ satisfies the defining relation of the Weyl group.
- For details see arXiv:0808.2604.

Quantum geometric crystal structure on $\mathcal{A}_q$

- The algebra homomorphism $\mathbf{e}_i^t : Q(\mathcal{A}_q) \to Q(\mathcal{A}_q)(t)$ is defined by

$$\mathbf{e}_i^t(e_j) = e_i^n e_j e_i^{-n}\big|_{q_i^n \mapsto t} \qquad (j \in I),$$
$$\mathbf{e}_i^t(q^\lambda) = t^{-\langle h_i, \lambda\rangle} q^\lambda \qquad (\lambda \in P).$$

- Then $\mathbf{e}_i^1 = \mathrm{id}_{Q(\mathcal{A}_q)}$, $\mathbf{e}_i^{t_1}\mathbf{e}_i^{t_2} = \mathbf{e}_i^{t_1 t_2} : Q(\mathcal{A}_q) \to Q(\mathcal{A}_q)(t_1, t_2)$.

- Furthermore $\{\mathbf{e}_i^t\}_{i \in I}$ satisfies the Verma relations:

  - $(a_{ij}, a_{ji}) = (0,0) \implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1}$,
  - $(a_{ij}, a_{ji}) = (-1,-1) \implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1 t_2}\mathbf{e}_i^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1 t_2}\mathbf{e}_j^{t_1}$,
  - $(a_{ij}, a_{ji}) = (-1,-2) \implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1 t_2}\mathbf{e}_i^{t_1 t_2^2}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1 t_2^2}\mathbf{e}_j^{t_1 t_2}\mathbf{e}_i^{t_1}$,
  - $(a_{ij}, a_{ji}) = (-1,-3)$

$$\implies \mathbf{e}_i^{t_1}\mathbf{e}_j^{t_1 t_2}\mathbf{e}_i^{t_1^2 t_2^3}\mathbf{e}_j^{t_1 t_2^2}\mathbf{e}_i^{t_1 t_2^3}\mathbf{e}_j^{t_2} = \mathbf{e}_j^{t_2}\mathbf{e}_i^{t_1 t_2^3}\mathbf{e}_j^{t_1 t_2^2}\mathbf{e}_i^{t_1^2 t_2^3}\mathbf{e}_j^{t_1 t_2}\mathbf{e}_i^{t_1}.$$

Definition of quantum geometric crystal

Definition. $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I})$ is called a **quantum geometric crystal** if it satisfies the following conditions:

- $\mathcal{K}$ is a skew field.
- $\mathbf{e}_i^t$ is an algebra homomorphism $\mathcal{K} \to \mathcal{K}(t)$.
- $\mathbf{e}_i^t$ is regular at $t = 1$. $\mathbf{e}_i^1 = \mathrm{id}_{\mathcal{K}}$, $\mathbf{e}_i^{t_1}\mathbf{e}_i^{t_2} = \mathbf{e}_i^{t_1 t_2}$.
- $\{\mathbf{e}_i^t\}_{i\in I}$ satisfies the Verma relations.
- $\mathbb{F}[q^P]$ is a subalgebra of the center of $\mathcal{K}$.
- $\mathbf{e}_i^t$ is regular at $t = q^\lambda$ for any $\lambda \in P$.
- $\mathbf{e}_i^t(q^\lambda) = t^{-\langle h_i, \lambda\rangle} q^\lambda$ for $\lambda \in P$. □

Remark. For the classical case, see Berenstein-Kazhdan math/9912105.

Proposition 2. $(Q(\mathcal{A}_q), \{\mathbf{e}_i^t\}_{i\in I})$ is a quantum geometric crystal. □

Weyl group action on a quantum geometric crystal

Proposition 3.

Let $(\mathcal{K}, \{\mathbf{e}_i^t\}_{i\in I})$ be a quantum geometric crystal.

Put $a_i = q^{\alpha_i} = q_i^{\alpha_i^\vee}$ and $s_i(x) = \mathbf{e}_i^{a_i}(x)$ for $i \in I$, $x \in \mathcal{K}$.

Then s_i is an algebra automorphism of $\mathcal{K}$ with

$$s_i(q^\lambda) = q^{\lambda - \langle h_i, \lambda\rangle \alpha_i} = q^{s_i(\lambda)} \qquad \text{for } \lambda \in P.$$

Moreover $\{s_i\}_{i\in I}$ satisfies the defining relations of the Weyl group W and hence generates the action of W on $\mathcal{K}$. □

- Propositions 2 and 3 $\Longrightarrow$ Theorem 1.

Quantum Schubert cell

- $b_{ij} := d_i a_{ij}$. Then $b_{ji} = b_{ij}$ and $q^{b_{ij}} = q_i^{a_{ij}}$.
- $\mathbf{i} := (i_1, i_2, \dots, i_N) \in I^N$.
- $\mathcal{A}_{\mathbf{i},0} :=$ the associative algebra over $\mathbb{F} = \mathbb{Q}(q)$ generated by $\{x_\nu\}_{\nu=1}^N$ with defining relations: $x_\nu x_\mu = q^{b_{i_\mu i_\nu}} x_\mu x_\nu \quad (\mu < \nu)$.
- $\mathcal{A}_{\mathbf{i}} := \mathbb{F}[q^P] \otimes_{\mathbb{F}} \mathcal{A}_{\mathbf{i},0} = \langle\, q^\lambda, x_\nu \mid \lambda \in P,\ 1 \leqq \nu \leqq N \,\rangle$.
  (Identification. $q^\lambda \otimes 1 = q^\lambda$, $1 \otimes x_\nu = x_\nu$)
- Then $\mathcal{A}_{\mathbf{i}}$ is an Ore domain.
- If $w = s_{i_1} s_{i_2} \cdots s_{i_N}$ is a reduced expression of $w \in W$, then $Q(\mathcal{A}_{0,\mathbf{i}})$ depends only on w (Berenstein q-alg/9605016) and is the rational function field of a **quantum Schubert cell**.

Quantum geometric crystal structure on $\mathcal{A}_{\mathbf{i}}$

- $e_i := \sum_{i_\nu = i} x_\nu$. Then $\{e_i\}_{i \in I}$ satisfies the q-Serre relations.
- Assume $\{ i_\nu \mid \nu = 1 \ldots, N \} = I$. ($\longleftarrow$ inessential assumtion)
  Then $e_i \neq 0$ for all $i \in I$.

Theorem 4. (quant. geom. crys. str. on $\mathcal{A}_i$)
The algebra hom. $\mathbf{e}_i^t : Q(\mathcal{A}_{\mathbf{i}}) \to Q(\mathcal{A}_{\mathbf{i}})(t)$ can be defined by

$$\mathbf{e}_i^t(x_\nu) = e_i^n x_\nu e_i^{-n}\Big|_{q_i^n \mapsto t}, \qquad \mathbf{e}_i(q^\lambda) = t^{-\langle h_i, \lambda \rangle} q^\lambda.$$

Then $(Q(\mathcal{A}_{\mathbf{i}}), \{\mathbf{e}_i\}_{i \in I})$ is a quantum geometric crystal. $\square$

Remark. An induction on $n = 0, 1, 2, \ldots$ proves that
$e_i^n x_\nu e_i^{-n}$ is an n-independent rational function of q_i^n.

Explicit formulae $\longrightarrow$ Next page

Explicit formulae for $\mathrm{e}_i^t(x_\nu)$ and their positivity

- $X := \sum_{i_\mu = i,\ \mu<\nu} x_\mu, \quad Y := \sum_{i_\mu = i,\ \mu>\nu} x_\mu.$.
- Then $e_i = X + \delta_{i_\nu i} x_\nu + Y$.
- If $i_\nu = i$, then

$$\mathbf{e}_i^t(x_\nu) = t^{-2} x_\nu \frac{1 + q_i^2 (x_\nu + q_i^2 Y) X^{-1}}{1 + q_i^2 t^{-2} (x_\nu + q_i^2 Y) X^{-1}} \frac{1 + q_i^2 Y (X + x_\nu)^{-1}}{1 + q_i^2 t^{-2} Y (X + x_\nu)^{-1}},$$

- If $i_\nu \neq i$, then $-a_{i i_\nu} \geqq 0$ and

$$\mathbf{e}_i^t(x_\nu) = t^{-a_{i i_\nu}} x_\nu \prod_{k=1}^{-a_{i i_\nu}} \frac{1 + q_i^{-2(k-1)} t^{-2} Y X^{-1}}{1 + q_i^{-2(k-1)} Y X^{-1}}.$$

Positivity. All the formulae for $\mathbf{e}_i^t(x_\nu)$ are **subtraction-free**.

Commentaries

- φ_i and ε_i for $\mathcal{A}_q$ and $\mathcal{A}_{\mathbf{i}}$ cases.
  - $\mathbf{e}_i^t(e_i) = e_i, \quad \mathbf{e}_i^t(q^{\alpha_i}) = t^{-2}q^{\alpha_i}$.
  - $\varepsilon_i :=$ const.$q^{\alpha_i}e_i, \quad \varphi_i :=$ const.$q^{-\alpha_i}e_i. \quad (\varepsilon_i = q^{2\alpha_i}\varphi_i)$
  - Then $\quad \mathbf{e}_i^t(\varphi_i) = t^2\varphi_i, \quad \mathbf{e}_i^t(\varepsilon_i) = t^{-2}\varepsilon_i$.
  - quantum t^{-2}, $q^{2\alpha_i} \longleftrightarrow$ classical "c", "α_i"

- Classical limit of $\mathcal{A}_{q,0}$:
  Poisson subvariety $X \subset U_-$.
  ($U_- = \exp \mathfrak{n}_-$, the lower maximal unipotent subgroup pf G)

- Classical limit of $\mathcal{A}_{\mathbf{i},0}$:
  $X_{\mathbf{i}} = \{(x_1, \dots, x_N)\} \to \{y_{i_1}(x_1) \cdots y_{i_N}(x_N)\} \subset U_-$.
  ($y_i(x) = \exp(xF_i)$, $F_i =$ (the lower Chevalley generator of $\mathfrak{g}$))

Files

- This file $\longrightarrow$
http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100924_Nagoya.pdf

- Quantum M-matrix for A_∞ case $\longrightarrow$ §1.6 of
http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100630_Osaka.pdf

- Quantization of the birational action of $W(A^{(1)}_{m-1}) \times W(A^{(1)}_{n-1})$ given by Kajiwara-Noumi-Yamada nlin/0106029 for mutually prime m, n
$\to$ http://www.math.tohoku.ac.jp/~kuroki/LaTeX/20100630_WxW.pdf

- Theory of quantum geometric crystals $\longrightarrow$ in preparation