Cycle theory of relative correspondences

Masaki Hanamura*

Abstract

We establish a theory of complexes of *relative correspondences*. The theory generalizes the known theory of complexes of correspondences of smooth projective varieties. It will be applied in the sequel of this paper to the construction of the triangulated category of motives over a base variety.

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We have the theory of algebraic correspondences of smooth projective varieties for the Chow group and for the higher Chow group. We first recall the classical theory of correspondences for the Chow group. For smooth projective varieties X, Y over a field k, let $\mathrm{CH}^r(X \times Y)$ be the Chow group of codimension r cycles of $X \times Y$. An element of this group is said to be a correspondence from X to Y. One has composition of correspondences defined as follows. Let Z be another smooth projective variety. For $u \in \mathrm{CH}^r(X \times Y)$ and $v \in \mathrm{CH}^s(Y \times Z)$, the composition $u \circ v \in \mathrm{CH}^{r+s-\dim Y}(X \times Z)$ is defined by

$$u \circ v = p_{13\,*}(p_{12}^* u \cdot p_{23}^* v)$$

where for example p_{12} is the projection from $X \times Y \times Z$ to $X \times Y$. One has associativity for composition: $(u \circ v) \circ w = u \circ (v \circ w)$. The theory of *motives* (to be precise Chow motives) over k is based on the theory of correspondences. The basic idea is to consider the additive category where objects are smooth projective varieties, morphisms are given by correspondences, and composition given by composition of correspondences.

Instead of the Chow group one can take the higher Chow group. For $u \in \mathrm{CH}^r(X \times Y, n)$ and $v \in \mathrm{CH}^s(Y \times Z, m)$ the composition $u \circ v \in \mathrm{CH}^{r+s-\dim Y}(X \times Z, n+m)$ is defined by the same formula. Indeed we can do this at the level of chain complexes. Recall for a variety X the cycle complex $(\mathcal{Z}^r(X, \cdot), \partial)$ is a chain complex where $\mathcal{Z}^r(X, n)$ is the free abelian group on the set of non-degenerate irreducible subvarieties V of $X \times \square^n$ meeting faces properly (see §0 for details). The boundary map ∂ is given by restricting cycles to codimension one faces and taking an alternating sum. The homology of this complex is the group $\mathrm{CH}^r(X, n)$. For X and Y smooth projective, $\mathcal{Z}^r(X \times Y, \cdot)$ is the complex of "higher" correspondences from X to Y. For $u \in \mathcal{Z}^r(X \times Y, n)$ and $v \in \mathcal{Z}^s(Y \times Z, m)$ the pull-backs $p_{12}^* u$ and $p_{23}^* v$ may not meet properly in $X \times Y \times Z \times \square^{n+m}$. But according to a moving lemma the subcomplex

$$\mathcal{Z}^r(X \times Y, \cdot) \hat{\otimes} \mathcal{Z}^s(Y \times Z, \cdot)$$

*Department of Mathematics, Tohoku University, Aramaki Aoba-Ku, 980-8587, Sendai, Japan

of $\mathcal{Z}^r(X \times Y, \cdot) \otimes \mathcal{Z}^s(Y \times Z, \cdot)$ generated by elements $u \otimes v$, where u, v are non-degenerate irreducible subvarieties such that $p_{12}^* u$ and $p_{23}^* v$ meet properly, is a quasi-isomorphic subcomplex. For such u and v, the composition $u \circ v \in \mathcal{Z}^{r+s-\dim Y}(X \times Z, \cdot)$ is defined, yielding a map of complexes

$$\rho : \mathcal{Z}^r(X \times Y, \cdot) \hat{\otimes} \mathcal{Z}^s(Y \times Z, \cdot) \to \mathcal{Z}^{r+s-\dim Y}(X \times Z, \cdot) \ .$$

If W is a fourth smooth projective variety, the subcomplex $\mathcal{Z}(X \times Y, \cdot) \hat{\otimes} \mathcal{Z}(Y \times Z, \cdot) \hat{\otimes} \mathcal{Z}(Z \times W, \cdot)$, generated by $u \otimes v \otimes w$ such that the triple $p_{12}^* u, p_{23}^* v, p_{34}^* w$ is properly intersecting on the four-fold product, is a quasi-isomorphic subcomplex. For such u, v, w, one has $u \circ v \circ w \in \mathcal{Z}(X \times W, \cdot)$ defined by $p_{14*}(p_{12}^* u \cdot p_{23}^* v \cdot p_{34}^* w)$, and the following holds: $u \circ v \circ w = (u \circ v) \circ w = u \circ (v \circ w)$.

Complexes $\mathcal{Z}(X \times Y, \cdot)$ and the partially defined composition were used in the construction of a theory of the triangulated category of *mixed motives* over k, see [6]. An object of the category is *a diagram of smooth projective varieties* which consists of a sequence of smooth projective varieties and higher correspondences between them, subject to certain conditions.

We would like to generalize this to relative correspondences. Let S be a quasi-projective variety over k. By a *smooth variety* X *over* S we mean a smooth variety over k, equipped with a projective map to S (the map $X \to S$ need not be smooth). Let X and Y be smooth varieties over S. A natural choice for the complex of correspondences from X to Y would be $\mathcal{Z}_a(X \times_S Y, \cdot)$, the cycle complex of dimension a cycles of the fiber product $X \times_S Y$. Since the variety $X \times_S Y$ is not smooth, we need to replace this with another complex of abelian groups $F(X, Y)$. Concretely $F(X, Y)$ is the cone of the restriction map of the cycle complexes $\mathcal{Z}(X \times Y, \cdot) \to \mathcal{Z}(X \times Y - X \times_S Y, \cdot)$, shifted by -1. Even after replacing it with $F(X, Y)$, there is no partially defined composition map. What we can achieve is the following.

(1) There is a complex $F(X, Y)$ and an injective quasi-isomorphism of complexes $\mathcal{Z}(X \times_S Y, \cdot) \to F(X, Y)$. To be precise one should keep track of the dimensions of the cycle complex, which we ignore now.

(2) If Z is another smooth variety, projective over S, there is a quasi-isomorphic subcomplex

$$\iota : F(X, Y) \hat{\otimes} F(Y, Z) \hookrightarrow F(X, Y) \otimes F(Y, Z) \ .$$

(3) There is another complex $F(X, Y, Z)$ and a surjective quasi-isomorphism

$$\sigma : F(X, Y, Z) \to F(X, Y) \hat{\otimes} F(Y, Z) \ .$$

(4) There is a map of complexes $\varphi : F(X, Y, Z) \to F(X, Z)$.
In the derived category at least, one has an induced map $F(X, Y) \otimes F(Y, Z) \to F(X, Z)$ obtained by composing ι^{-1}, σ^{-1}, and φ. This map plays the role of composition. One should note, in contrast to the case $S = \operatorname{Spec} k$, there is no composition map defined on $F(X, Y) \hat{\otimes} F(Y, Z)$; the composition φ is defined only on $F(X, Y, Z)$.

The pattern persists for more than three varieties. For the formulation it is convenient to change the notation as follows. In the above situation, write X_1, X_2 and X_3 in place of X, Y, Z; let

$$F(X_1, X_2, X_3 \top \{2\}) := F(X_1, X_2) \otimes F(X_2, X_3)$$

and

$$F(X_1, X_2, X_3 | \{2\}) := F(X_1, X_2) \hat{\otimes} F(X_2, X_3) \ .$$

Then the maps are of the form $\iota_2 : F(X_1, X_2, X_3 \top \{2\}) \hookrightarrow F(X_1, X_2, X_3 | \{2\})$, $\sigma_2 : F(X_1, X_2, X_3) \to F(X_1, X_2, X_3 \mid \{2\})$, and $\varphi_2 : F(X_1, X_2, X_3) \to F(X_1, X_3)$.

The generalization goes as follows.

(1) For each sequence of smooth varieties over S, $X_1, \cdots X_n$ $(n \geq 2)$, there corresponds a complex $F(X_1, \cdots, X_n)$. If $n = 2$ there is an injective quasi-isomorphism $\mathcal{Z}(X_1 \times_S X_2, \cdot) \to F(X_1, X_2)$.

For a subset of integers $S = \{i_1, \cdots, i_{a-1}\} \subset (1, n)$, let $i_0 = 1$, $i_a = n$ and

$$F(X_1, \cdots, X_n \,\overline{|}\, S) := F(X_{i_0}, \cdots, X_{i_1}) \otimes F(X_{i_1}, \cdots, X_{i_2}) \otimes \cdots \otimes F(X_{i_{a-1}}, \cdots, X_{i_a}) \ .$$

There is a complex $F(X_1, \cdots, X_n|S)$ and an injective quasi-isomorphism

$$\iota_S : F(X_1, \cdots, X_n|S) \hookrightarrow F(X_1, \cdots, X_n \,\overline{|}\, S) \ .$$

We assume $F(X_1, \cdots, X_n|\emptyset) = F(X_1, \cdots, X_n)$.

(2) For $S \subset S'$ there is a surjective quasi-isomorphism

$$\sigma_{S\,S'} : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, X_n|S') \ .$$

For $S \subset S' \subset S''$, $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. In particular we have $\sigma_S := \sigma_{\emptyset\,S} : F(X_1, \cdots, X_n) \to F(X_1, \cdots, X_n|S)$.

(3) For $K = \{k_1, \cdots, k_b\} \subset (1, n)$ disjoint from S, a map

$$\varphi_K : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n|S) \ .$$

If K is the disjoint union of K' and K'', one has $\varphi_K = \varphi_{K'}\varphi_{K''}$.

(4) If K and S' are disjoint $\sigma_{S\,S'}$ and φ_K commute.

Indeed there is a more precise description. Each complex $F(X_1, \cdots, X_n)$ is a degreewise free $\mathbb{Z}$-module on a given set of generators. In the situation of (1), for a set of generators

$$\alpha_k \in F(X_{i_{k-1}}, \cdots, X_{i_k}) \quad k = 1, \cdots, a-1 \ ,$$

there is a condition whether the set is *properly intersecting*. The $F(X_1, \cdots, X_n|S)$ is the subcomplex generated by $\alpha_1 \otimes \cdots \otimes \alpha_{a-1}$ for properly intersecting tuples $\alpha_1, \cdots, \alpha_{a-1}$. In particular it is a multiple subcomplex of $F(X_1, \cdots, X_n \,\overline{|}\, S)$. For the full details and additional properties see §2.

The description of $F(X_1, \cdots, X_n \,\overline{|}\, S)$ in terms of properly intersecting sets may seem excess baggage. In order to describe variants of such subcomplexes, however, it is necessary to utilize the notion of properly intersecting sets. To illustrate this by a simple example, let $n < m$ and given a sequence of varieties $X_1, \cdots, X_m$, a subset $S \subset (1, n)$, and an element $f \in F(X_n, \cdots, X_m)$. The subcomplex of $F(X_1, \cdots, X_n|S)$ generated by $\alpha_1 \otimes \cdots \otimes \alpha_{a-1}$ such that $\{\alpha_1, \cdots, \alpha_{a-1}, f\}$ is properly intersecting is a quasi-isomorphic subcomplex. This subcomplex is denoted $[F(X_1, \cdots, X_n|S)]_f$ and called the distinguished subcomplex with respect to the constraint f. The full argument on variations of such subcomplexes can be found in §3.

In §1 and 2, we define the complexes $F(X_1, \cdots, X_n)$ as above for a sequence of smooth quasi-projective varieties $X_1, \cdots, X_n$, each equipped with a projective map to a base variety S. We now explain the ideas for the construction in case $n \leq 3$.

In §1, given a smooth variety M and a finite ordered open covering $\mathcal{U}$ of an open set $U \subset M$, we define a complex $\mathcal{Z}(M, \mathcal{U})$ which is quasi-isomorphic to the cycle complex $\mathcal{Z}(A, \cdot)$

of $A = M - U$. If $\mathcal{U} = \{U\}$, the covering consisting of U only, $\mathcal{Z}(M, \mathcal{U})$ is the cone of the restriction map $\mathcal{Z}(M) \to \mathcal{Z}(U)$, shifted by -1. In general one replaces $\mathcal{Z}(U)$ by $\mathcal{Z}(\mathcal{U})$, the Čech complex with respect to the covering.

Assume M' is another smooth variety, $\mathcal{U}'$ a finite ordered open covering of $U' \subset M'$; assume also there are smooth maps $q : M \to Y$ and $q' : M' \to Y$. Let $M \times_Y M'$ be the fiber product and $p : M \times_Y M' \to M$, $p' : M \times_Y M' \to M'$ be the projections. One has a covering $p^{-1}\mathcal{U} \amalg p'^{-1}\mathcal{U}'$ of the open set $p^{-1}U \cup p'^{-1}U'$ of $M \times_Y M'$. For $u \in \mathcal{Z}(M, \mathcal{U})$ and $v \in \mathcal{Z}(M', \mathcal{U}')$ one has the pull-backs $p^*u \in \mathcal{Z}(M \times_Y M', p^{-1}\mathcal{U})$ and $p'^*v \in \mathcal{Z}(M \times_Y M', p'^{-1}\mathcal{U}')$, and if they meet properly, their product is defined as an element of $\mathcal{Z}(M \times_Y M', p^{-1}\mathcal{U} \amalg p'^{-1}\mathcal{U}')$. The subcomplex $\mathcal{Z}(M, \mathcal{U}) \hat{\otimes} \mathcal{Z}(M', \mathcal{U}') \subset \mathcal{Z}(M, \mathcal{U}) \otimes \mathcal{Z}(M', \mathcal{U}')$ generated by such $u \otimes v$ is shown to be a quasi-isomorphic subcomplex, and the product gives a map of complexes

$$\rho : \mathcal{Z}(M, \mathcal{U}) \hat{\otimes} \mathcal{Z}(M', \mathcal{U}') \to \mathcal{Z}(M \times_Y M', p^{-1}\mathcal{U} \amalg p'^{-1}\mathcal{U}') \ .$$

If $p : M \to N$ is a projective map, $\mathcal{V}$ a covering of an open set of $V \subset N$, then $p^{-1}\mathcal{V}$ is an open covering of $p^{-1}V \subset M$, and there is the projection map $p_* : \mathcal{Z}(M, p^{-1}\mathcal{V}) \to \mathcal{Z}(N, \mathcal{V})$.

If we apply this to $A = X \times_S Y \subset M = X \times Y$ and the covering consisting only of $U_{12} := M - A$, one obtains a complex $\mathcal{Z}(X \times Y, \{U_{12}\})$. If we set $F(X, Y)$ to be this complex our problem is partially solved. If Z is another variety over S, one has $F(Y, Z) = \mathcal{Z}(Y \times Z, \{U_{23}\})$ with $U_{23} = Y \times Z - Y \times_S Z$, and there is the product map

$$\rho : \mathcal{Z}(X \times Y, \{U_{12}\}) \hat{\otimes} \mathcal{Z}(Y \times Z, \{U_{23}\}) \to \mathcal{Z}(X \times Y \times Z, \{p_{12}^{-1}(U_{12}), p_{23}^{-1}(U_{23})\}) \ .$$

The problem remains, since from the target of ρ there is no projection p_{13*} to the cycle complex $\mathcal{Z}(X \times Z, \{U_{13}\})$ where $U_{13} = X \times Z - X \times_S Z$.

One notices here that there is a restriction map

$$r : \mathcal{Z}(X \times Y \times Z, \{U_{123}\}) \to \mathcal{Z}(X \times Y \times Z, \{p_{12}^{-1}(U_{12}), p_{23}^{-1}(U_{23})\}) \ ,$$

where $U_{123} = X \times Y \times Z - X \times_S Y \times_S Z$, since U_{123} contains both $p_{12}^{-1}(U_{12})$ and $p_{23}^{-1}(U_{23})$. The map r is a quasi-isomorphism, since both complexes are quasi-isomorphic to $\mathcal{Z}(X \times_S Y \times_S Z)$. Assume for simplicity Y is projective. One then defines the projection along p_{13} as the composition

$$p_{13*} : \mathcal{Z}(X \times Y \times Z, \{U_{123}\}) \to \mathcal{Z}(X \times Y \times Z, \{p_{13}^{-1}U_{13}\}) \to \mathcal{Z}(X \times Z, \{U_{13}\}) \ .$$

Here the first map is the restriction, which is defined since $U_{123} \supset p_{13}^{-1}U_{13}$, and the second map is the projection along p_{13}. Consider now the double complex

$$\begin{array}{ccc} & & \mathcal{Z}(X \times Y, \{U_{12}\}) \hat{\otimes} \mathcal{Z}(Y \times Z, \{U_{23}\}) \\ & & \Big\downarrow \rho \\ \mathcal{Z}(X \times Y \times Z, \{U_{123}\}) & \xrightarrow{r} & \mathcal{Z}(X \times Y \times Z, \{p_{12}^{-1}(U_{12}), p_{23}^{-1}(U_{23})\}) \end{array}$$

where the upper right corner and lower left corner are placed in degree 0, and let $F(X, Y, Z)$ be the total complex. In other words it is the cone of $r + \rho$ shifted by -1. The required properties are satisfied with this. The map $\sigma : F(X, Y, Z) \to F(X, Y) \hat{\otimes} F(Y, Z)$ is given by the projection to $\mathcal{Z}(X \times Y, \{U_{12}\}) \hat{\otimes} \mathcal{Z}(Y \times Z, \{U_{23}\})$, the map $\varphi : F(X, Y, Z) \to F(X, Z)$ is obtained by composing the projection to $\mathcal{Z}(X \times Y \times Z, \{U_{123}\})$ with the map p_{13*}.

The construction of the complexes $F(X_1, \cdots, X_n)$ for $n \geq 3$ and the maps σ, φ consists of a systematic generalization of the above. In §1 we discuss the properties of the complexes

$\mathcal{Z}(M,\mathcal{U})$ and their tensor products. In §2 we construct the complexes $F(X_1,\cdots,X_n|S)$ and the maps ι, σ, and φ. The construction uses a variant of the so-called *bar complex*. Since this construction appears again in a different context in Part II, we give an axiomatic description.

In §4 we construct the *diagonal cycles* which play the role of the identity. Let $\Delta_X \in \mathcal{Z}(X\times_S X,0)$ be the element given by the diagonal $X\subset X\times_S X$. Its image under the inclusion to $F(X,X)$ is also denoted Δ_X; it has degree 0 and boundary zero. One can construct, for $n\geq 2$, an element $\boldsymbol{\Delta}_X(1,\cdots,n)\in F(\overbrace{X,\cdots,X}^{n})$ of degree 0 with boundary zero, satisfying the properties below. For the statement we introduce some notation. When X is understood, for any subset $I=\{j_1,\cdots,j_m\}\subset[1,n]$ set $F(I)=F(\overbrace{X,\cdots,X}^{m})$ and $\boldsymbol{\Delta}_X(I)=\boldsymbol{\Delta}_X(j_1,\cdots,j_m)\in F(I)$. For $S\subset(1,n)$ let $\tau_S: F(X_1,\cdots,X_n)\to F(X_1,\cdots,X_n\,\rceil S)$ be the composition of σ_S and ι_S.

(1) One has $\boldsymbol{\Delta}_X(1,2)=\Delta_X\in F(X,X)$.

(2) If $S=\{i_1,\cdots,i_{a-1}\}\subset(1,n)$, and $I_1,\cdots,I_{a-1}$ the corresponding segmentation, one has

$$\tau_S(\boldsymbol{\Delta}_X(1,\cdots,n)\,)=\boldsymbol{\Delta}(I_1)\otimes\cdots\otimes\boldsymbol{\Delta}(I_{a-1})$$

in $F(X,\cdots,X\,\rceil S)=F(I_1)\otimes\cdots\otimes F(I_{a-1})$.

(3) For $K\subset(1,n)$, $\varphi_K(\boldsymbol{\Delta}(1,\cdots,n)\,)=\boldsymbol{\Delta}([1,n]-K)$.

We then show the existence of "diagonal extensions". To explain it in the simplest case, let $n<m$, and assume given a sequence of varieties X_i on $[1,n]$. Setting $X_i=X_n$ for $i\in[n,m]$ we extend the sequence to $[1,m]$. On $[n,m]$ one has a constant sequence, so there is the diagonal cycle $\boldsymbol{\Delta}([n,m])\in F([n,m])=F(X_n,\cdots,X_n)$. Recall the map $\tau_n: F(X_1,\cdots,X_m)\to F(X_1,\cdots,X_n)\otimes F([n,m])$. There is then a map of complexes called the diagonal extension

$$\mathrm{diag}: F(X_1,\cdots,X_n)\to F(X_1,\cdots,X_m)$$

such that $\tau_n\,\mathrm{diag}: F(X_1,\cdots,X_n)\to F(X_1,\cdots,X_n)\otimes F([n,m])$ coincides with $u\mapsto u\otimes\boldsymbol{\Delta}([n,m])$. In other words, $\mathrm{diag}(u)$ is a canonical lifting of $u\otimes\boldsymbol{\Delta}([n,m])$ with respect to τ_n. The map diag is also compatible with the maps φ.

The constructions and results in Part I show that the classes of smooth varieties over S, the complexes $F(X_1,\cdots,X_n)$ and the maps σ, φ form a *quasi DG category*. To be more specific, a symbol over S is a formal finite sum $\bigoplus_\alpha(X_\alpha/S,r_\alpha)$ where X_α is a smooth variety over S and $r_\alpha\in\mathbb{Z}$. To a finite sequence of symbols $K_1,\cdots,K_n$ ($n\geq 2$) and a subset $S\subset(1,n)$ one can associate a complex of abelian groups $F(K_1,\cdots,K_n|S)$; if $K_i=(X_i,r_i)$, then $F(K_1,\cdots,K_n)$ is the complex $F(X_1,\cdots,X_n|S)$, the integers r_i specifying the dimensions of the cycle complexes involved. One has maps $\sigma_{S\,S'}$ and φ_K for $F(K_1,\cdots,K_n|S)$ as well. The class of symbols over S, the complexes $F(K_1,\cdots,K_n|S)$, the maps $\sigma_{S\,S'}$, φ_K, along with additional structure – generating set for the complex, notion of properly intersecting elements, distinguished subcomplexes with respect to constraints, diagonal cycles and diagonal extension – constitute a quasi DG category.

In the sequel of this paper we introduce the notion of quasi DG category, which is a generalization of DG category. A quasi DG category consists of a class of objects, complexes $F(X_1,\cdots,X_n|S)$ for a sequence of objects, maps $\sigma_{S\,S'}$, φ_K and additional structure that are subject to a set of axioms. The axioms is an abstraction of the properties verified for the relative cycle complexes.

In the section titled "Basic notions" we have collected materials needed throughout the paper.

Contents.

§0. Basic notions.
§1. The Čech cycle complexes $\mathcal{Z}(M, \mathcal{U})$.
§2. Function complexes $F(X_1, \cdots, X_n)$.
§3. Distinguished subcomplexes with respect to a constraint.
§4. The diagonal cycle and the diagonal extension.

0 Basic notions

(0.1) *The cycle complex.* In this paper k is an arbitrary ground field, and one considers separated schemes of finite type (we will simply say schemes) over k. A variety is a reduced, possibly reducible scheme over k.

The references for the cycle complex are [1], [2], [3]. We briefly recall some definitions and results that will be needed in this paper.

Let $\square^1 = \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \cdots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square^{n-1}_{i,a} = \{x_i = a\}$ where $a = 0$ or ∞. A face of dimension m is canonically isomorphic to $\square^m$.

Let X be an equi-dimensional variety (or a scheme). Let $\mathcal{Z}^r(X \times \square^n)$ be the free abelian group on the set of codimension r irreducible subvarieties of $X \times \square^n$ meeting each $X \times$ face properly. An element of $\mathcal{Z}^r(X \times \square^n)$ is called an *admissible* cycle. The inclusions of codimension one faces $\delta_{i,a} : \square^{n-1}_{i,a} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^i (\delta^*_{i,0} - \delta^*_{i,\infty}) : \mathcal{Z}^r(X \times \square^n) \to \mathcal{Z}^r(X \times \square^{n-1}) .$$

One has $\partial \circ \partial = 0$. Let $\pi_i : X \times \square^n \to X \times \square^{n-1}$, $i = 1, \cdots, n$ be the projections, and $\pi_i^* : \mathcal{Z}^r(X \times \square^{n-1}) \to \mathcal{Z}^r(X \times \square^n)$ be the pull-backs. Let $\mathcal{Z}^r(X, n)$ be the quotient of $\mathcal{Z}^r(X \times \square^n)$ by the sum of the images of π_i^*. Thus an element of $\mathcal{Z}^r(X, n)$ is a represented uniquely by a cycle whose irreducible components are non-degenerate (not a pull-back by π_i). The map ∂ induces a map $\partial : \mathcal{Z}^r(X, n) \to \mathcal{Z}^r(X, n-1)$, and $\partial \circ \partial = 0$. The complex $\mathcal{Z}^r(X, \cdot)$ thus defined is the *cycle complex* of X in codimension r. The higher Chow groups are the homology groups of this complex:

$$\mathrm{CH}^r(X, n) = H_n \mathcal{Z}^r(X, \cdot) .$$

Note $\mathrm{CH}^r(X, 0) = \mathrm{CH}^r(X)$, the Chow group of X. In this paper we would rather use the indexing by dimensions: for $s \in \mathbb{Z}$, $\mathcal{Z}_s(X, \cdot) = \mathcal{Z}^{\dim X - r}(X, \cdot)$, and $\mathrm{CH}_s(X, n)$ is the homology group.

If X is a quasi-projective variety and U is an open set, letting $Z = X - U$, one has an exact sequence of complexes $0 \to \mathcal{Z}_s(Z, \cdot) \to \mathcal{Z}_s(X, \cdot) \to \mathcal{Z}_s(U, \cdot)$. The localization theorem [2]says that the induced map $\mathcal{Z}_s(X, \cdot)/\mathcal{Z}_s(Z, \cdot) \to \mathcal{Z}_s(U, \cdot)$ is a quasi-isomorphism (indeed a homotopy equivalence).

A proper map $f : X \to Y$ gives rise to a map of complexes $f_* : \mathcal{Z}_s(X, \cdot) \to \mathcal{Z}_s(Y, \cdot)$. A flat map $f : X \to Y$ of dimension d induces a map of complexes $f^* : \mathcal{Z}_s(Y, \cdot) \to \mathcal{Z}_{s+d}(X, \cdot)$. There is also a partially defined pull-back map f^* associated with a map $f : X \to Y$, if Y is smooth. See [3].

There is a result called the "easy moving lemma" in [3]; a generalization of this lemma will be discussed in §1.

(0.2) *Multiple complexes.* By a complex of abelian groups we mean a graded abelian group $A^\bullet$ with a map d of degree one satisfying $dd = 0$. If $u : A \to B$ and $v : B \to C$ are maps of complexes, we define $u \cdot v : A \to C$ by $(u \cdot v)(x) = v(u(x))$. So $u \cdot v$ is $v \circ u$ in the usual notation. As usual we also write vu for $v \circ u$ (but not for $v \cdot u$).

A double complex $A = (A^{i,j}; d', d'')$ is a bi-graded abelian group with differentials d' of degree $(1, 0)$, d'' of degree $(0, 1)$, satisfying $d'd'' + d''d' = 0$. Its total complex $\mathrm{Tot}(A)$ is the complex with $\mathrm{Tot}(A)^k = \bigoplus_{i+j=k} A^{i,j}$ and the differential $d = d' + d''$. In contrast a "double" complex $A = (A^{i,j}; d', d'')$ is a bi-graded abelian group with differentials d' of degree $(1, 0)$, d'' of degree $(0, 1)$, satisfying $d'd'' = d''d'$. Its total complex $\mathrm{Tot}(A)$ is given by $\mathrm{Tot}(A)^k = \bigoplus_{i+j=k} A^{i,j}$ and the differential $d = d' + (-1)^i d''$ on $A^{i,j}$.

Let (A, d_A) and (B, d_B) be complexes. Then $(A^{i,j} = A^j \otimes B^i; 1 \otimes d_B, d_A \otimes 1)$ is a "double" complex; notice the first grading comes from the grading of B. Its total complex has differential d given by

$$d(x \otimes y) = (-1)^{\deg y} dx \otimes y + x \otimes dy \ .$$

Note this differs from the usual convention.

More generally $n \geq 2$ one has the notion of n-tuple complex and "n-tuple" complex. An n-tuple (resp. "n-tuple") complex is a $\mathbb{Z}^n$-graded abelian group $A^{i_1,\cdots,i_n}$ with differentials $d_1, \cdots, d_n$, d_k raising i_k by 1, such that for $k \neq \ell$, $d_k d_\ell + d_\ell d_k = 0$ (resp. $d_k d_\ell = d_\ell d_k$). A single complex $\mathrm{Tot}(A)$, called the total complex, is defined in either case. As a variant one can define partial totalization: For a subset $S = [k, \ell] \subset [1, n]$ with cardinality ≥ 2, one can "totalize" in degrees in S, so the result $\mathrm{Tot}^S(A)$ is an m-tuple (resp. "m-tuple") complex, where $m = n - |S| + 1$.

For n complexes $A_1^\bullet, \cdots, A_n^\bullet$, the tensor product $A_1^\bullet \otimes \cdots \otimes A_n^\bullet$ is an "n-tuple" complex.

The difference between n-tuple and "n-tuple" complexes is slight, so we often do not make the distinction. There is an obvious notion of maps of n-tuple ("n-tuple") complexes.

If A is an n-tuple complex and B an m-tuple complex, and when $S = [k, \ell] \subset [1, n]$ with $m = n - |S| + 1$ is specified, one can talk of maps of m-tuple complexes $\mathrm{Tot}^S(A) \to B$. When the choice of S is obvious from the context, we just say maps of multiple complexes $A \to B$. For example if A is an n-tuple complex and B an $(n-1)$-tuple complex, for each set $S = [k, k+1]$ in $[1, n]$ one can speak of maps of $(n-1)$-tuple complexes $\mathrm{Tot}^S(A) \to B$; if $n = 2$ there is no ambiguity.

(0.2.1) *Multiple subcomplexes of a tensor product complex.* Let A and B be complexes. A double subcomplex $C^{i,j} \subset A^i \otimes B^j$ is a submodule closed under the two differentials. If $\mathrm{Tot}(C) \hookrightarrow \mathrm{Tot}(A \otimes B)$ is a quasi-isomorphism, we say $C^{\bullet\bullet}$ is a quasi-isomorphic subcomplex. It is convenient to let $A^\bullet \hat{\otimes} B^\bullet$ denote such a subcomplex. (Note it does not mean the tensor product of subcomplexes of A and B.) Likewise a quasi-isomorphic multiple subcomplex of $A_1^\bullet \otimes \cdots \otimes A_n^\bullet$ is denoted $A_1^\bullet \hat{\otimes} \cdots \hat{\otimes} A_n^\bullet$.

(0.3) *Tensor product of "double" complexes.* Let $A^{\bullet,\bullet} = (A^{a,p}; d'_A, d''_A)$ be a "double" complex (so d' has degree $(1, 0)$, d'' has degree $(0, 1)$, and $d'd' = 0$, $d''d'' = 0$ and $d'd'' = d''d'$). The associated total complex $\mathrm{Tot}(A)$ has differential d_A given by $d_A = d' + (-1)^a d''$ on $A^{a,p}$. The association $A \mapsto \mathrm{Tot}(A)$ forms a functor. Let $(B^{b,q}; d'_B, d''_B)$ be another "double" complex. Then the tensor product of A and B as "double" complexes, denoted $A^{\bullet,\bullet} \times B^{\bullet,\bullet}$, is by definition the "double" complex $(E^{c,r}; d'_E, d''_E)$, where

$$E^{c,r} = \bigoplus_{a+b=c\,,p+q=r} A^{a,p} \otimes B^{b,q}$$

and $d'_E = (-1)^b d'_A \otimes 1 + 1 \otimes d'_B$, $d''_E = (-1)^q d''_A \otimes 1 + 1 \otimes d''_B$.

The tensor product complex $\mathrm{Tot}(A) \otimes \mathrm{Tot}(B)$ and the total complex of $A^{\bullet,\bullet} \times B^{\bullet,\bullet}$ are related as follows. There is an isomorphism of complexes

$$u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(A^{\bullet,\bullet} \times B^{\bullet,\bullet})$$

given by $u = (-1)^{aq} \cdot id$ on the summand $A^{a,p} \otimes B^{b,q}$.

Let A, B, C be "double" complexes. One has an obvious isomorphism of "double" complexes $(A \times B) \times C = A \times (B \times C)$; it is denoted $A \times B \times C$. The following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \otimes \mathrm{Tot}(C) & \xrightarrow{u\otimes 1} & \mathrm{Tot}(A \times B) \otimes \mathrm{Tot}(C) \\
{\scriptstyle 1\otimes u}\downarrow & & \downarrow{\scriptstyle u} \\
\mathrm{Tot}(A) \otimes \mathrm{Tot}(B \times C) & \xrightarrow{u} & \mathrm{Tot}(A \times B \times C)\ .
\end{array}$$

The composition defines an isomorphism $u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \otimes \mathrm{Tot}(C) \xrightarrow{\sim} \mathrm{Tot}(A \times B \times C)$.

One can generalize this to the case of tensor product of more than two "double" complexes. If $A_1, \cdots, A_n$ are "double" complexes, there is an isomorphism of complexes

$$u_n : \mathrm{Tot}(A_1) \otimes \cdots \otimes \mathrm{Tot}(A_n) \to \mathrm{Tot}(A_1 \times \cdots \times A_n)$$

which coincides with the above u if $n = 2$, and is in general a composition of u's in any order. As in case $n = 3$, one has commutative diagrams involving u's; we leave the details to the reader.

Let A, B, C be "double" complexes and $\rho : A^{\bullet,\bullet} \times B^{\bullet,\bullet} \to C^{\bullet,\bullet}$ be a map of "double" complexes, namely it is bilinear and for $\alpha \in A^{a,p}$ and $\beta \in B^{b,q}$,

$$d'\rho(\alpha \otimes \beta) = \rho((-1)^b d'\alpha \otimes \beta + \alpha \otimes d'\beta)$$

and

$$d''\rho(\alpha \otimes \beta) = \rho((-1)^q d''\alpha \otimes \beta + \alpha \otimes d''\beta)\ .$$

Composing $\mathrm{Tot}(\rho) : \mathrm{Tot}(A \times B) \to \mathrm{Tot}(C)$ with $u : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \xrightarrow{\sim} \mathrm{Tot}(A \times B)$, one obtains the map

$$\hat{\rho} : \mathrm{Tot}(A) \otimes \mathrm{Tot}(B) \to \mathrm{Tot}(C)\ ;$$

it is given given by $(-1)^{aq} \cdot \rho$ on the summand $A^{a,p} \otimes B^{b,q}$.

The same holds for a map of "double" complexes $\rho : A_1 \times \cdots \times A_n \to C$.

(0.4) *The bar complex* (§2). Let (A, d_A) be a differential graded algebra, namely a complex of abelian groups with associative multiplication, satisfying $d(\alpha \cdot \beta) = (-1)^{\deg \beta}(d\alpha) \cdot \beta + \alpha \cdot (d\beta)$. (Usually one considers a differential graded algebra with augmentation, and take A to be its augmentation ideal.) The bar complex $B(A)$ is, as an abelian group, the tensor algebra (over $\mathbb{Z}$) $T(A) = \bigoplus_{c\geq 0} A^{\otimes c}$.

Give a grading by

$$\overline{\deg}(\alpha_1 \otimes \cdots \otimes \alpha_c) = \sum(\deg \alpha_i - 1)$$

and give a pair of differentials by (put $\epsilon_j = \deg(\alpha_j) - 1$)

$$\bar{d}(\alpha_1 \otimes \cdots \otimes \alpha_c) = -\sum(-1)^{\sum_{j>i}\epsilon_j} \alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes d_A(\alpha_i) \otimes \cdots \otimes \alpha_c \ ,$$

$$\bar{\rho}(\alpha_1 \otimes \cdots \otimes \alpha_c) = \sum(-1)^{\sum_{j\geq i}\epsilon_j} \alpha_1 \otimes \cdots \otimes \alpha_{i-2} \otimes (\alpha_{i-1} \cdot \alpha_i) \otimes \cdots \otimes \alpha_c \ .$$

Since $\bar{d}\bar{d} = 0$, $\bar{\rho}\bar{\rho} = 0$ and $\bar{d}\bar{\rho} + \bar{\rho}\bar{d} = 0$ as can be verified, $d_{B(A)} = \bar{d} + \bar{\rho}$ is a differential. The bar complex is the complex with the grading and the differential $d_{B(A)}$. There is a filtration by subcomplexes of $B(A)$ so that the successive quotients are

$$A[1] \otimes \cdots \otimes A[1] \quad (c \text{ times })$$

as complexes.

(0.5) *Finite ordered sets, partitions and segmentations.* Let I be a non-empty finite totally ordered set (we will simply say a finite ordered set), so $I = \{i_1, \cdots, i_n\}, i_1 < \cdots < i_n$, where $n = |I|$. The *initial* (resp. *terminal*) element of I is i_1 (resp. i_n); let $\mathrm{in}(I) = i_1$, $\mathrm{tm}(I) = i_n$. If $n \geq 2$, let $\overset{\circ}{I} = I - \{\mathrm{in}(I), \mathrm{tm}(I)\}$.

If $I = \{i_1, \cdots, i_n\}$, a subset I' of the form $[i_a, i_b] = \{i_a, \cdots, i_b\}$ is called a *sub-interval.*

In the main body of the paper, for the sake of concreteness we often assume $I = [1, n] = \{1, \cdots, n\}$, a subset of $\mathbb{Z}$. More generally a finite subset of $\mathbb{Z}$ is an example of a finite ordered set.

A *partition* of I is a disjoint decomposition into sub-intervals $I_1, \cdots, I_a$ such that there is a sequence of integers $i < i_1 < \cdots < i_{a-1} < j$ so that $I_k = [i_{k-1}, i_k - 1]$, with $i_0 = i$ and $i_a = j + 1$.

So far we have assumed I and I_i to be of cardinality ≥ 1. In some contexts we allow only finite ordered sets with at least two elements. There instead of partition the following notion plays a role. Given a subset of $\overset{\circ}{I}$, $\Sigma = \{i_1, \cdots, i_{a-1}\}$, where $i_1 < i_2 < \cdots < i_{a-1}$, one has a decomposition of I into the sub-intervals $I_1, \cdots, I_a$, where $I_k = [i_{k-1}, i_k]$, with $i_0 = i_1$, $i_a = i_n$. Thus the sub-intervals satisfy $I_k \cap I_{k+1} = \{i_k\}$ for $k = 1, \cdots, a - 1$. The sequence of sub-intervals $I_1, \cdots, I_a$ is called the *segmentation* of I corresponding to Σ. (The terminology is adopted to distinguish it from the partition).

Finite ordered sets of cardinality ≥ 1 and partitions appear in connection with a sequence of fiberings. On the other hand, finite ordered sets of cardinality ≥ 2 and segmentations appear when we consider a sequence of varieties (or an associated sequence of fiberings). See below.

(0.6) *Sequence of fiberings* (§1). Let $n \geq 2$. A *sequence of fiberings* consists of smooth varieties M_i ($1 \leq i \leq n$) and Y_i ($1 \leq i \leq n - 1$), together with smooth maps $M_i \to Y_i$ and $M_{i+1} \to Y_i$. For a sub-interval $I = [j, k] \subset [1, n]$ of cardinality ≥ 1, one defines M_I to be the fiber product $M_j \times_{Y_j} M_{j+1} \times \cdots \times_{Y_{k-1}} M_k$. If $I_1, \cdots, I_c$ is a partition of $[1, n]$, then one has smooth varieties $M_{I_1}, \cdots, M_{I_c}$, which form a sequence of smooth varieties over appropriate Y's.

(0.7)*Sequence of varieties* (§2). Let $n \geq 2$. A *sequence of smooth varieties* over S is a set of smooth varieties X_i indexed by $i \in [1, n]$, where each X_i is equipped with a projective map to

S. For a sub-interval $I = [j,k]$ of cardinality ≥ 2, let X_I be the direct product $\prod_{i\in I} X_i$. Given an segmentation $I_1, \cdots, I_c$ corresponding to $\Sigma = \{i_k\}$, the varieties X_{I_t} and the projections to X_{i_k} form a sequence of fiberings.

(0.8) *The class of symbols over S.* Let S be a quasi-projective variety. Let (Smooth$/k$, Proj$/S$) be the category of smooth varieties X equipped with projective maps to S. A *symbol* over S is an object the form

$$\bigoplus_{\alpha\in A} (X_\alpha/S, r_\alpha)$$

where X_α is a collection of objects in (Smooth$/k$, Proj$/S$) indexed by a finite set A, and $r_\alpha \in \mathbb{Z}$. The class of objects over S is denoted $Symb(S)$.

1 The Čech cycle complexes $\mathcal{Z}(M,\mathcal{U})$

Let k be a fixed ground field. By a smooth variety over k we mean a smooth quasi-projective equi-dimensional variety over k.

(1.1) *I-coverings.* Let M be a smooth variety over k, $A \subset M$ a closed set, and $U := M - A$. Let I be a finite ordered set. An open covering of U indexed by I (or just an *I-covering* of U) is a set of open sets $\mathcal{U} = \{U_i\}_{i\in I}$, with $\cup_i U_i = U$. It is also denoted by $(I,\mathcal{U})$.

If $V \subset M$ is another open set, J is another finite ordered set and $\mathcal{V} = \{V_j\}_{j\in J}$ a J-covering of V, a map of coverings $(I,\mathcal{U}) \to (J,\mathcal{V})$, or just $\mathcal{U} \to \mathcal{V}$ for short, is an order preserving map $\lambda : J \to I$ such that $U_{\lambda(j)} \supset V_j$ for $j \in J$. One then has $V \subset U$. We thus have the category of I-coverings of open sets of M, for varying I; it is denoted by $Cov(\mathcal{O}(M))$. The subcategory of I-coverings of a given $U \subset M$ is denoted $Cov(U \subset M)$, or just $Cov(U)$.

If $\mathcal{U}$ is an I-covering of U and $\lambda : J \to I$ is an order preserving map, define $\lambda^*\mathcal{U}$ to be the J-covering of $U' = \cup_j U_{\lambda(j)}$ given by $j \mapsto U_{\lambda(j)}$. There is a natural map of coverings $\lambda^* : (I,\mathcal{U}) \to (J,\lambda^*\mathcal{U})$. For composition of maps λ, $\lambda^*\mathcal{U}$ is contravariant functorial. A map of coverings $\lambda : (I,\mathcal{U}) \to (J,\mathcal{V})$ factors as $(I,\mathcal{U}) \xrightarrow{\lambda^*} (J,\lambda^*\mathcal{U}) \to (J,\mathcal{V})$.

If $\mathcal{U}$ is an I-covering of U and $\mathcal{U}'$ an I'-covering of U' then one has an $I \amalg I'$-covering $\mathcal{U} \amalg \mathcal{U}'$ of $U \cup U'$. Here $I \amalg I'$ is ordered so that $i < i'$ for $i \in I$ and $i' \in I'$.

For the rest of this section, without so mentioning an indexing set I is always finite ordered, and a map between them is always order preserving.

The notion of coverings and maps can be defined for I unordered or infinite. For our purposes we restrict to finite ordered indexing sets.

(1.2) *The complex $\mathcal{Z}(M,\mathcal{U})$.* For a variety X let $\mathcal{Z}_s(X,\cdot)$ denote the cubical cycle complex as in [3]; an element of the complex is uniquely represented by an admissible cycle on $X \times \square^n$, whose components are non-degenerate. We have the cycle complex $\mathcal{Z}_s(U,\cdot)$, $s \in \mathbb{Z}$ for an open set $U \subset M$. We will abbreviate it to $\mathcal{Z}_s(U)$, or $\mathcal{Z}(U)$. From now we often drop the dimension s from the notation, as long as there is no confusion.

For an I-covering $\mathcal{U}$ of U, we define a complex denoted $\mathcal{Z}(M,\mathcal{U})$ to be the total complex associated to the double complex $A^{\bullet,\bullet}$ defined as follows. Let

$$A^{a,0} = \mathcal{Z}(M, -a) \ ,$$

and for $p \geq 0$,

$$A^{a,p+1} = \bigoplus_{i_0 < \cdots < i_p} \mathcal{Z}(U_{i_0,\cdots,i_p}, -a)$$

where $U_{i_0,\cdots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. It is convenient to set $U_\emptyset = M$, and when $p = -1$, we interpret $(i_0, \cdots, i_p) = \emptyset$, so $\alpha = (\alpha_\emptyset) \in \mathcal{Z}(M)$. With this convention, the differential δ of degree $(0,1)$ is given by sending $\alpha \in \bigoplus \mathcal{Z}(U_{i_0,\cdots,i_p})$ to

$$\delta(\alpha)_{i_0,\cdots,i_{p+1}} = \sum (-1)^{p-r+1} \alpha_{i_0,\cdots,\widehat{i_r},\cdots,i_{p+1}} | U_{i_0,\cdots,i_{p+1}} \ .$$

(The sign differs from the usual sign convention of Čech complexes.) The differential ∂ of degree $(1,0)$ is the boundary map of each cycle complex. The differential of the total complex is $d = \partial + (-1)^a \delta$ on $A^{a,p}$.

$$\mathcal{Z}_s(M,\mathcal{U}) = \left[\mathcal{Z}(M) \xrightarrow{\delta} \bigoplus \mathcal{Z}(U_{i_0}) \xrightarrow{\delta} \bigoplus_{i_0<i_1} \mathcal{Z}(U_{i_0\, i_1}) \to \cdots \to \bigoplus_{i_0<\cdots<i_p} \mathcal{Z}(U_{i_0,\cdots,i_p}) \to \cdots \right] .$$

Note the natural map

$$\iota : \mathcal{Z}_s(A) \to \mathcal{Z}_s(M,\mathcal{U})$$

is a quasi-isomorphism. This follows from the localization theorem for the cycle complex [2].

If $(J,\mathcal{V})$ covers V, and $\lambda : (I,\mathcal{U}) \to (J,\mathcal{V})$ a map of coverings, there is the induced map of complexes

$$\mathcal{Z}(M,\lambda) : \mathcal{Z}(M,\mathcal{U}) \to \mathcal{Z}(M,\mathcal{V}) \ ;$$

thus we have a functor $\mathcal{Z}(M,-)$ from the category $Cov(\mathcal{O}(M))$ to $C(Ab)$. The following square commutes:

$$\begin{array}{ccc} \mathcal{Z}(M,\mathcal{U}) & \xrightarrow{\mathcal{Z}(M,\lambda)} & \mathcal{Z}(M,\mathcal{V}) \\ \uparrow & & \uparrow \\ \mathcal{Z}(A) & \longrightarrow & \mathcal{Z}(B) \ . \end{array}$$

Here $B = M - V$, and the bottom is the map induced by inclusion.

As special cases of $\mathcal{Z}(M,\lambda)$, we have restriction maps and pull-backs (in general, a map $\mathcal{Z}(M,\lambda)$ between cycle complexes is a composition of restriction and pull-back).

If $I = J$ and $\mathcal{U}$, $\mathcal{V}$ are coverings such that $V_i \subset U_i$, there is the *restriction map*

$$\mathcal{Z}(M,\mathcal{U}) \to \mathcal{Z}(M,\mathcal{V}) \ .$$

If $\lambda : J \to I$ is a map and $\mathcal{V} = \lambda^*\mathcal{U}$, then $\lambda^* : (I,\mathcal{U}) \to (J,\lambda^*\mathcal{U})$ induces a map (called *pull-back*)

$$\lambda^* : \mathcal{Z}(M,\mathcal{U}) \to \mathcal{Z}(M,\lambda^*\mathcal{U}) \ .$$

(1.3) *Push-forward and pull-back.* If $p : M \to N$ is a projective map, $B \subset N$ a closed set with complement V such that $p^{-1}V = U$ and $\mathcal{V} \in Cov(V \subset N)$, then $p^{-1}\mathcal{V} \in Cov(U \subset M)$ is defined in the obvious manner, and push-forward on cycle complexes induces a map (also called the push-forward)

$$p_* : \mathcal{Z}_s(M, p^{-1}\mathcal{V}) \to \mathcal{Z}_s(N,\mathcal{V}) \ .$$

It is compatible with $p_* : \mathcal{Z}_s(A) \to \mathcal{Z}_s(B)$ via the maps ι.

The push-forward is covariant functorial in p. It commutes with the maps of functoriality for coverings: For a map $\lambda : (I, \mathcal{U}) \to (J, \mathcal{V})$ of coverings in $Cov(\mathcal{O}(N))$, one has the induced map of coverings $\lambda : (I, p^{-1}\mathcal{U}) \to (J, p^{-1}\mathcal{V})$ in $Cov(\mathcal{O}(M))$, and the following square commutes:

$$\begin{array}{ccc}
\mathcal{Z}(M, p^{-1}\mathcal{U}) & \xrightarrow{\mathcal{Z}(M,\lambda)} & \mathcal{Z}(M, p^{-1}\mathcal{V}) \\
{\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p_*} \\
\mathcal{Z}(N, \mathcal{U}) & \xrightarrow{\mathcal{Z}(N,\lambda)} & \mathcal{Z}(N, \mathcal{V}) \ .
\end{array}$$

Let $p : M \to N$ be a smooth map of relative dimension d. For $\mathcal{V} \in Cov(V \subset N)$, the pull-back map

$$p^* : \mathcal{Z}_s(N, \mathcal{V}) \to \mathcal{Z}_{s+d}(M, p^{-1}\mathcal{V})$$

is defined. It is compatible with the pull-back map

$$p^* : \mathcal{Z}_s(B) \to \mathcal{Z}_{s+d}(p^{-1}B)$$

(B is the complement of V) via the maps ι. The pull-back is contravariant functorial in p. It commutes with functoriality maps for coverings $\mathcal{V}$.

(1.4) *Restricted tensor product and the product map.* Let M, M' and Y be smooth varieties and $q : M \to Y$, $q' : M' \to Y$ be *smooth* maps of varieties. Let $M \diamond M' := M \times_Y M'$ and $p : M \diamond M' \to M$, $p' : M \diamond M' \to M'$ be the projections.

$$\begin{array}{ccccc}
 & & M \diamond M' & & \\
 & \swarrow & & \searrow & \\
M & & & & M' \\
 & \searrow & & \swarrow & \\
 & & Y & &
\end{array}$$

Let $a, b \in \mathbb{Z}$, and $c = a + b - \dim Y$. We define a subcomplex

$$\mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M') \subset \mathcal{Z}_a(M) \otimes \mathcal{Z}_b(M')$$

as follows. An element $\alpha \otimes \beta \in \mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M')$ (α and β are assumed to be irreducible) is in the subcomplex iff $p^*\alpha$ and $p'^*\beta$ meet properly in $M \diamond M'$, and the product $p^*\alpha \cdot p'^*\beta \in \mathcal{Z}_c(M \diamond M')$. We define

$$\alpha \circ_Y \beta = \alpha \circ \beta := p^*\alpha \cdot p'^*\beta \ .$$

(We say briefly that the condition is that $\alpha \circ \beta \in \mathcal{Z}(M \diamond M')$ be defined.)

Then the following conditions are satisfied; (i) is non-trivial and proved later in this section, and the rest are immediate from the definitions.

(i) The inclusion of the subcomplex is a quasi-isomorphism.

(ii) There is a map of complexes

$$\rho_Y = \rho : \mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M') \to \mathcal{Z}_c(M \diamond M')$$

which sends $\alpha \otimes \beta$ to $\alpha \circ_Y \beta$.

(iii) If $\pi : N \to M$ is a smooth map of dimension d, the pull-back $\pi^* \otimes id : \mathcal{Z}_a(M) \otimes \mathcal{Z}_b(M') \to \mathcal{Z}_{a+d}(N) \otimes \mathcal{Z}_b(M')$ takes the subcomplex $\mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M')$ into $\mathcal{Z}_{a+d}(N) \hat{\otimes} \mathcal{Z}_b(M')$. We thus have the induced map

$$\pi^* \otimes id : \mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M') \to \mathcal{Z}_{a+d}(N) \hat{\otimes} \mathcal{Z}_b(M') \ .$$

Similar property holds for pull-backs in M'. This applies in particular to open immersions into M or M'.

If $\pi : M \to N$ is a projective map, the push-forward $\pi_* \otimes id : \mathcal{Z}_a(M) \otimes \mathcal{Z}_b(M') \to \mathcal{Z}_a(N) \otimes \mathcal{Z}_b(M')$ takes the subcomplex $\mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M')$ into $\mathcal{Z}_a(N) \hat{\otimes} \mathcal{Z}_b(M')$:

$$\pi_* \otimes id : \mathcal{Z}_a(M) \hat{\otimes} \mathcal{Z}_b(M') \to \mathcal{Z}_a(N) \hat{\otimes} \mathcal{Z}_b(M') \ .$$

Similar property holds for push-forward in M'.

To state the next level of generalization, let $M_1 \to Y_1 \leftarrow M_2 \to Y_2 \leftarrow M_3$ be a sequence of smooth varieties and smooth maps. We have $M_i \diamond M_{i+1} = M_i \times_{Y_i} M_{i+1}$ as before, and $M_1 \diamond M_2 \diamond M_3 = M_1 \times_{Y_1} M_2 \times_{Y_2} M_3$. Note $M_1 \diamond M_2 \diamond M_3 = (M_1 \diamond M_2) \diamond M_3 = M_1 \diamond (M_2 \diamond M_3)$. Let $p_i : M_1 \diamond M_2 \diamond M_3 \to M_i$ be the projection.

$$\begin{tikzcd}
 & & M_1 \diamond M_2 \diamond M_3 \arrow[dl] \arrow[dr] & & \\
 & M_1 \diamond M_2 \arrow[dl] \arrow[dr] & & M_2 \diamond M_3 \arrow[dl] \arrow[dr] & \\
M_1 \arrow[dr] & & M_2 \arrow[dl] \arrow[dr] & & M_3 \arrow[dl] \\
 & Y_1 & & Y_2 &
\end{tikzcd}$$

Define a subcomplex

$$\mathcal{Z}(M_1) \hat{\otimes} \mathcal{Z}(M_2) \hat{\otimes} \mathcal{Z}(M_3) \subset \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3)$$

as follows. (In what follows we will not specify the dimensions a_i for $\mathcal{Z}_{a_i}(M_i)$.) An element $\alpha_1 \otimes \alpha_2 \otimes \alpha_3 \in \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3)$ (α_i are assumed irreducible) is in the subcomplex iff the following conditions (i)-(iii) are satisfied:

(i) $p_1^*\alpha_1$ and $p_2^*\alpha_2$ meet properly in $M_1 \diamond M_2 \diamond M_3$, and $p_1^*\alpha_1 \cdot p_2^*\alpha_2 \in \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$. (This is equivalent to an analogous condition for the pull-backs of α_1 and α_2 to $M_1 \diamond M_2$.)

(ii) $p_2^*\alpha_2$ and $p_3^*\alpha_3$ meet properly in $M_1 \diamond M_2 \diamond M_3$, and $p_2^*\alpha_2 \cdot p_3^*\alpha_3 \in \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$.

(iii) $p_1^*\alpha_1 \cdot p_2^*\alpha_2$ and $p_3^*\alpha_3$ meet properly in $M_1 \diamond M_2 \diamond M_3$, and $(p_1^*\alpha_1 \cdot p_2^*\alpha_2) \cdot p_3^*\alpha_3 \in \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$.

Similarly $p_1^*\alpha_1$ and $p_2^*\alpha_2 \cdot p_3^*\alpha_3$ and meet properly in $M_1 \diamond M_2 \diamond M_3$, and $p_1^*\alpha_1 \cdot (p_2^*\alpha_2 \cdot p_3^*\alpha_3) \in \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$. Note then the two triple intersections coincide.

By (iii) one can define

$$\alpha_1 \underset{Y_1}{\circ} \alpha_2 \underset{Y_2}{\circ} \alpha_3 := p_1^*\alpha_1 \cdot p_2^*\alpha_2 \cdot p_3^*\alpha_3 \in \mathcal{Z}(M_1 \diamond M_2 \diamond M_3).$$

By (i) and (iii), $\alpha_1 \circ \alpha_2 \in \mathcal{Z}(M_1 \diamond M_2)$ and $(\alpha_1 \circ \alpha_2, \alpha_3) \in \mathcal{Z}(M_1 \diamond M_2)\hat{\otimes}\mathcal{Z}(M_3)$. Similarly $\alpha_2 \circ \alpha_3 \in \mathcal{Z}(M_2 \diamond M_3)$ and $(\alpha_1, \alpha_2 \circ \alpha_3) \in \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2 \diamond M_3)$. Further,

$$\alpha_1 \circ \alpha_2 \circ \alpha_3 = (\alpha_1 \circ \alpha_2) \circ \alpha_3 = \alpha_1 \circ (\alpha_2 \circ \alpha_3) \ .$$

The following statements hold, the only non-trivial one being the first assertion in (i).

(i) The inclusion of the subcomplex is a quasi-isomorphism. There are also inclusions

$$\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \subset (\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)) \otimes \mathcal{Z}(M_3)$$

and

$$\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \subset \mathcal{Z}(M_1) \otimes (\mathcal{Z}(M_2))\hat{\otimes}\mathcal{Z}(M_3)) \ .$$

(ii) There is a map of complexes

$$\rho_{Y_1} : \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \to \mathcal{Z}(M_1 \diamond M_2)\hat{\otimes}\mathcal{Z}(M_3)$$

which sends $(\alpha_1, \alpha_2, \alpha_2)$ to $(\alpha_1 \circ \alpha_2, \alpha_3)$, a similar map

$$\rho_{Y_2} : \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \to \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2 \diamond M_3) \ ,$$

and

$$\rho_{Y_1Y_2} : \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \to \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$$

which sends $(\alpha_1, \alpha_2, \alpha_3)$ to $\alpha_1 \circ \alpha_2 \circ \alpha_3$. One has $\rho_{Y_1Y_2} = \rho_{Y_2}\rho_{Y_1}$, where ρ_{Y_2} is the product map $\mathcal{Z}(M_1 \diamond M_2)\hat{\otimes}\mathcal{Z}(M_3) \to \mathcal{Z}(M_1 \diamond M_2 \diamond M_3)$. Similarly $\rho_{Y_1Y_2} = \rho_{Y_1}\rho_{Y_2}$.

We may shorten the notation and write ρ_i for ρ_{Y_i} and ρ_{12} for $\rho_{Y_1Y_2}$. We may alternatively write $\rho([1,2],[3])$ for ρ_{Y_1}, $\rho([1],[2,3])$ for ρ_{Y_2}, and $\rho([1,3])$ for $\rho_{Y_1Y_2}$. Here $[j,k]$ denotes the set of integers between j and k.

The inclusions and the product maps are compatible in the sense that the following square commutes:

$$\begin{array}{ccc}
\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) & \xrightarrow{\rho_{Y_1}} & \mathcal{Z}(M_1 \diamond M_2)\hat{\otimes}\mathcal{Z}(M_3) \\
{\scriptstyle \mathrm{incl}}\downarrow & & {\scriptstyle \mathrm{incl}}\downarrow \\
\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3) & \xrightarrow{\rho_{Y_1}\otimes 1} & \mathcal{Z}(M_1 \diamond M_2) \otimes \mathcal{Z}(M_3) \ .
\end{array}$$

Similarly for ρ_{Y_2}.

(iii) If $\pi : N_1 \to M_1$ is a smooth map of dimension d, the pull-back

$$\pi^* \otimes 1 \otimes 1 : \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3) \to \mathcal{Z}(N_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3)$$

takes the subcomplex $\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3)$ into $\mathcal{Z}(N_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3)$. So the map $\pi^* \otimes 1 \otimes 1 : \mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3) \to \mathcal{Z}(N_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3)$ is defined. Similar property in each M_i.

If $\pi : M_1 \to N_1$ is a projective map, the push-forward

$$\pi_* \otimes 1 \otimes 1 : \mathcal{Z}(M_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3) \to \mathcal{Z}(N_1) \otimes \mathcal{Z}(M_2) \otimes \mathcal{Z}(M_3)$$

takes the $\mathcal{Z}(M_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3)$ into $\mathcal{Z}(N_1)\hat{\otimes}\mathcal{Z}(M_2)\hat{\otimes}\mathcal{Z}(M_3)$. Similar property in each M_i.

The general formulation, which involves n varieties, goes as follows. Let $n \geq 2$, and assume given smooth varieties M_i $(1 \leq i \leq n)$ and Y_i $(1 \leq i \leq n-1)$ with smooth maps $M_i \to Y_i$ and $M_{i+1} \to Y_i$. (We call such data a *sequence of fiberings* indexed by $[1, n]$.)

$$\begin{tikzcd}
M_1 \arrow[dr] & & M_2 \arrow[dl] \arrow[dr] & & \cdots \arrow[dr] & & M_n \arrow[dl] \\
& Y_1 & & Y_2 & & Y_{n-1} &
\end{tikzcd}$$

For a sub-interval $I = [j, k] \subset [1, n]$, let

$$M_I = M_j \diamond M_{j+1} \diamond \cdots \diamond M_k \ .$$

There are projection maps $p_i : M_I \to M_i$ for each $i \in I$. One can define a subcomplex

$$\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n) \subset \mathcal{Z}(M_1) \otimes \cdots \otimes \mathcal{Z}(M_n)$$

so that properties analogous to (i)-(iii) above are satisfied. The details will be given later in (1.6)-(1.9). Here we only note that the product map is of the following form.

Let $I_1, \cdots, I_c$ be a partition of $[1, n]$ into sub-intervals, see §0. The varieties $M_{I_1}, \cdots, M_{I_c}$ form a sequence of fiberings. The product map is of the form

$$\rho(I_1, \cdots, I_c) : \mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n) \to \mathcal{Z}(M_{I_1}) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_{I_c}) \ ,$$

that sends $(\alpha_1, \cdots, \alpha_n)$ to $(\alpha_{I_1}, \cdots, \alpha_{I_c})$, where $\alpha_{I_i} = \alpha_j \circ \cdots \circ \alpha_k$ if $I_i = [j, k]$.

(1.5) *Product map between Čech cycle complexes.* Let M, M' and Y be as above. For open subsets $U \subset M$ and $U' \subset M'$, let $U \diamond U' = U \times_Y U' = p^{-1}(U) \cap p'^{-1}(U') \subset M \diamond M'$. If A, A' are complements of U, U', $A \diamond A' := A \times_Y A'$ is the complement of $p^{-1}(U) \cup p'^{-1}(U')$.

Given coverings $\mathcal{U} \in Cov(U \subset M)$ and $\mathcal{V} \in Cov(U' \subset M')$, one defines a quasi-isomorphic subcomplex

$$\mathcal{Z}(M, \mathcal{U}) \hat{\otimes} \mathcal{Z}(M', \mathcal{U}') \subset \mathcal{Z}(M, \mathcal{U}) \otimes \mathcal{Z}(M', \mathcal{U}')$$

as the direct sum

$$\bigoplus \mathcal{Z}(U_{i_0, \cdots, i_p}) \hat{\otimes} \mathcal{Z}(V_{j_0, \cdots, j_q}) \subset \bigoplus \mathcal{Z}(U_{i_0, \cdots, i_p}) \otimes \mathcal{Z}(V_{j_0, \cdots, j_q}) \ .$$

Then one defines a map

$$\rho_Y = \rho : \mathcal{Z}(M, \mathcal{U}) \hat{\otimes} \mathcal{Z}(M', \mathcal{U}') \to \mathcal{Z}(M \diamond M', p^{-1}\mathcal{U} \amalg p'^{-1}\mathcal{U}')$$

which sends $\alpha \otimes \alpha' \in \mathcal{Z}(M, \mathcal{U}) \hat{\otimes} \mathcal{Z}(M', \mathcal{U}')$ to $\alpha \circ_Y \alpha'$ given by

$$(\alpha \circ_Y \alpha')_{i_0 \cdots i_p j_0 \cdots j_q} = \alpha_{i_0 \cdots i_p} \circ_Y \alpha'_{j_0 \cdots i_q} \ .$$

Here α consists of components $\alpha_{i_0 \cdots i_p} \in \mathcal{Z}(U_{i_0, \cdots, i_p})$, where we interpret $\alpha_\emptyset \in \mathcal{Z}(M)$ if $p = -1$. Recall $\mathcal{Z}(M, \mathcal{U})$ and $\mathcal{Z}(M', \mathcal{U}')$ are "double" complexes, so their tensor product may also be viewed as a "double" complex. One verifies that ρ_Y is a map of "double" complexes. The induced map of simple complexes, which sends $\alpha \otimes \alpha' \in \mathcal{Z}(U_{i_0, \cdots, i_p}, a) \hat{\otimes} \mathcal{Z}(V_{j_0, \cdots, j_q}, b)$ to $(-1)^{a(q+1)} \alpha \circ_Y \alpha'$, will also be denoted ρ_Y. (For the sign change, see (0.3)).

If $\lambda : \mathcal{U} \to \mathcal{V}$ and $\lambda' : \mathcal{U}' \to \mathcal{V}'$ are maps of coverings of open sets of M, M', respectively, there is an induced map of coverings $p^{-1}\mathcal{U} \amalg p'^{-1}\mathcal{U}' \to p^{-1}\mathcal{V} \amalg p'^{-1}\mathcal{V}'$, and one easily verifies that the following diagram commutes, either as a diagram of "double" complexes or as one of simple complexes.

$$\begin{array}{ccc} \mathcal{Z}(M,\mathcal{U})\hat{\otimes}\mathcal{Z}(M',\mathcal{U}') & \xrightarrow{\rho} & \mathcal{Z}(M\diamond M', p^{-1}\mathcal{U}\amalg p'^{-1}\mathcal{U}') \\ {\scriptstyle\lambda\otimes\lambda'}\downarrow & & \downarrow \\ \mathcal{Z}(M,\mathcal{V})\hat{\otimes}\mathcal{Z}(M',\mathcal{V}') & \xrightarrow{\rho} & \mathcal{Z}(M\diamond M', p^{-1}\mathcal{V}\amalg p'^{-1}\mathcal{V}') \end{array}$$

Since $A \diamond A'$ is the complement of $p^{-1}U \cup p'^{-1}U'$, the above ρ gives rise to a map in the derived category

$$\rho_Y : \mathcal{Z}_a(A) \otimes \mathcal{Z}_b(A') \to \mathcal{Z}_{a+b-\dim Y}(A \diamond A')$$

which makes the following diagram commute:

$$\begin{array}{ccccc} \mathcal{Z}(M,\mathcal{U})\otimes\mathcal{Z}(M',\mathcal{U}') & \hookleftarrow & \mathcal{Z}(M,\mathcal{U})\hat{\otimes}\mathcal{Z}(M',\mathcal{U}') & \xrightarrow{\rho} & \mathcal{Z}(M\diamond M', p^{-1}\mathcal{U}\amalg p'^{-1}\mathcal{U}') \\ {\scriptstyle\iota\otimes\iota}\uparrow & & & & \uparrow{\scriptstyle\iota} \\ \mathcal{Z}(A)\otimes\mathcal{Z}(A') & & \xrightarrow{\rho} & & \mathcal{Z}(A\diamond A')\ . \end{array}$$

All this can be generalized as follows. Let M_i $(1 \le i \le n)$ and Y_i $(1 \le i \le n-1)$ be smooth varieties with smooth maps $M_i \to Y_i$ and $M_{i+1} \to Y_i$, as before. Let $U_i \subset M_i$ be open sets, and $A_i = M_i - U_i$. For an interval $I = [j,k]$, one has M_I and the projections $p_i : M_I \to M_i$. The complement of the union of $p_i^{-1}U_i$ for $i \in I$ is $A_I = A_j \diamond \cdots \diamond A_k$. If $\mathcal{U}_i \in Cov(U_i \subset M_i)$, then one can define a quasi-isomorphic subcomplex

$$\mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) \subset \mathcal{Z}(M_1,\mathcal{U}_1)\otimes\cdots\otimes\mathcal{Z}(M_n,\mathcal{U}_n)\ .$$

For a partition $I_1, \cdots I_r$ of $[1,n]$, let

$$\mathcal{U}_{I_a} = \coprod_{i\in I_a} p_i^{-1}\mathcal{U}_i\ ,$$

a covering of $\cup_{i\in I_a}(p_i^{-1}U_i) \subset M_{I_a}$. We have a map

$$\rho(I_1,\cdots,I_r) : \mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) \to \mathcal{Z}(M_{I_1},\mathcal{U}_{I_1})\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_{I_r},\mathcal{U}_{I_r})$$

(put appropriate signs as in the case $n = 2$).

See (1.9) for a continuation of this subsection.

(1.6) *Distinguished subcomplexes of cycle complexes.* In [3]Bloch showed, for a smooth variety X, the subcomplex of $\mathcal{Z}(X,\cdot)$ consisting of the cycles meeting a given set of subvarieties of X properly is a quasi-isomorphic subcomplex. We discuss a generalization of this.

Let X be a smooth quasi-projective variety. A finite set $\{\alpha_1,\cdots,\alpha_n\}$ of irreducible subvarieties of X is *properly intersecting* if for any subset $\{i_1,\cdots,i_r\}$ of $\{1,\cdots,n\}$, the intersection $\alpha_{i_1}\cap\cdots\cap\alpha_{i_r}$ is empty or has codimension equal to the sum of the codimensions of α_{i_k}. A set of cycles $\{\alpha_1,\cdots,\alpha_n\}$ is properly intersecting if for all irreducible components of α_i's the above condition is satisfied. Then the intersection cycle $\alpha_{i_1}\cdot\alpha_{i_2}\cdot\cdots\cdot\alpha_{i_r}$ is well-defined, independent of the order of taking intersections.

Let $X_1, \cdots, X_r$, and T be a set of smooth quasi-projective varieties, and $W = \{W_\lambda\}$ be a finite set of admissible cycles of $X_1 \times \cdots \times X_r \times T \times \square^{\ell_\lambda}$ (*admissible* means meeting faces properly). Let $\pi_i : X_1 \times \cdots \times X_r \times T \to X_i$ be the projection. Let $s_1, \cdots, s_r$ be a sequence of integers. We have the cycle complexes $\mathcal{Z}_{s_i}(X_i)$ and their tensor product $\mathcal{Z}_{s_1}(X_1) \otimes \cdots \otimes \mathcal{Z}_{s_r}(X_r)$. From now we will usually drop the dimensions from the notation.

We define the subcomplex (called the *distinguished subcomplex* with respect to T and W)

$$[\mathcal{Z}(X_1) \otimes \cdots \otimes \mathcal{Z}(X_r)]_W \subset \mathcal{Z}(X_1) \otimes \cdots \otimes \mathcal{Z}(X_r)$$

as the subgroup generated by elements

$$\alpha_1 \otimes \cdots \otimes \alpha_r \in \mathcal{Z}(X_1, n_1) \otimes \cdots \otimes \mathcal{Z}(X_r, n_r)$$

where α_i are irreducible non-degenerate subvarieties satisfying the following condition:

(PI) For each λ, the set of cycles

$$\{\pi_1^*\alpha_1, \cdots, \pi_r^*\alpha_r, W_\lambda, \text{faces}\}$$

is properly intersecting in $X_1 \times \cdots \times X_r \times T \times \square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda}$. Here we employ the following obvious abuse of notation:

- π_i denotes also the projection $X_1 \times \cdots \times X_r \times T \times \square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda} \to X_i \times \square^{n_i}$;
- W_λ denotes its pull-back by the projection $X_1 \times \cdots \times X_r \times T \times \square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda} \to X_1 \times \cdots \times X_r \times T \times \square^{\ell_\lambda}$;
- a face is a closed set of the form $X_1 \times \cdots \times X_r \times T \times F$ where F is a face of $\square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda}$.

By a *distinguished subcomplex* we mean the distinguished subcomplex with respect to some T and W.

Theorem. *The inclusion $[\mathcal{Z}(X_1) \otimes \cdots \otimes \mathcal{Z}(X_r)]_W \subset \mathcal{Z}(X_1) \otimes \cdots \otimes \mathcal{Z}(X_r)$ is a quasi-isomorphism.*

This is proved in case X_i are smooth projective in [6], Part II, generalizing [3].

The case X_i are smooth quasi-projective is similar. We sketch here an argument communicated to us by M. Levine. Assume $r = 1$, so we must show $\mathcal{Z}_W(X) \hookrightarrow \mathcal{Z}(X)$ is a quasi-isomorphism. Take a projective closure $\bar{X}$ of X, let $Z = \bar{X} - X$, and consider the following commutative diagram:

$$\begin{array}{ccccc}
\mathcal{Z}(Z) & \longrightarrow & \mathcal{Z}(\bar{X}) & \longrightarrow & \mathcal{Z}(X) \\
\| & & \uparrow & & \uparrow \\
\mathcal{Z}(Z) & \longrightarrow & \mathcal{Z}_W(\bar{X}) & \longrightarrow & \mathcal{Z}_W(X) \; .
\end{array}$$

By the localization theorem [2]we know the map $\mathcal{Z}(\bar{X})/\mathcal{Z}(Z) \to \mathcal{Z}(X)$ is a quasi-isomorphism. The same proof shows $\mathcal{Z}_W(\bar{X})/\mathcal{Z}(Z) \to \mathcal{Z}_W(X)$ is a quasi-isomorphism. The argument in [3]shows $\mathcal{Z}_W(\bar{X}) \to \mathcal{Z}(\bar{X})$ is a quasi-isomorphism; although $\bar{X}$ is singular, the same proof works since W is contained in the smooth locus of $\bar{X}$. Hence one obtains the conclusion. We leave the case $r \geq 2$ to the reader.

Remarks. (1) The intersection of a finite number of distinguished subcomplexes is a distinguished subcomplex.

(2) For simplicity the defining condition (PI) may be phrased as follows, dropping π^* and $\square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda}$: The set $\{\alpha_1, \cdots, \alpha_r, W_\lambda, \text{faces}\}$ is properly intersecting in $X_1 \times \cdots \times X_r \times T$.

(3) The condition (PI) is equivalent to: For each λ and each face F of $\square^{n_1+\cdots+n_r} \times \square^{\ell_\lambda}$, the set of cycles $\{\alpha_1, \cdots, \alpha_r, W_\lambda \cap F\}$ is properly intersecting in $X_1 \times \cdots \times X_r \times T$. It follows from the following lemma.

(4) All the distinguished subcomplexes in the sequel of this paper are of the type in Example below.

Lemma. *Let X be a smooth variety, $\alpha_1, \cdots, \alpha_n$ be cycles on X, and $z_1, \cdots, z_m$ be properly intersecting cycles on X. Then the following are equivalent:*

(i) $\{\alpha_1, \cdots, \alpha_n, z_1, \cdots, z_m\}$ is properly intersecting in X.

(ii) For each intersection $z_{j_1} \cap \cdots \cap z_{j_p}$, where $1 \le j_1 < j_2 < \cdots < j_p \le m$, the set

$$\{\alpha_1, \cdots, \alpha_n, z_{j_1} \cap \cdots \cap z_{j_p}\}$$

is properly intersecting in X.

(1.6.1)**Example.** We will often see the following type of subcomplexes. Let $\{V_1, \cdots, V_k\}$ be a finite set of admissible cycles V_j on $X_1 \times \cdots \times X_r \times T \times \square^{\ell_j}$. We assume the set

$$\{V_1, \cdots, V_k, \text{ faces}\}$$

is *properly intersecting*. Consider the subcomplex generated by elements

$$\alpha_1 \otimes \cdots \otimes \alpha_r \in \mathcal{Z}(X_1, n_1) \otimes \cdots \otimes \mathcal{Z}(X_r, n_r)$$

where α_i are irreducible non-degenerate subvarieties satisfying the following condition:

$$\{\pi_1^*\alpha_1, \cdots, \pi_r^*\alpha_r, V_1, \cdots, V_k, \text{faces}\}$$

is properly intersecting in $X_1 \times \cdots \times X_r \times T \times \square^*$. Then it is a distinguished subcomplex. [Note it differs from the distinguished subcomplex with respect to $\{V_1, \cdots, V_k\}$, in which the proper intersection property is required with respect to each of V_j separately.]

To verify this assertion, one takes as the set W the collection of the partial intersections $V_1, \cdots, V_k$, and apply the above lemma.

(1.7) *The complex $\mathcal{Z}(M_1)\hat{\otimes} \cdots \hat{\otimes}\mathcal{Z}(M_n)$.* Assume given a sequence of fiberings, namely

($\star$) Smooth varieties M_i ($i = 1, \cdots, n$) and Y_i ($1 \le i \le n-1$), and smooth maps $M_i \to Y_i$ and $M_{i+1} \to Y_i$.

For an interval $I = [j, k] \subset [1, n]$, let

$$M_I = M_j \diamond M_{j+1} \diamond \cdots \diamond M_k \ .$$

There are projection maps $p_{I,i} : M_I \to M_i$ for each $i \in I$. More generally, if $I \subset I'$, there is the projection $p_{I',I} : M_{I'} \to M_I$, which is smooth. If $I = [1, n]$, $M_{[1,n]} = M_1 \diamond \cdots \diamond M_n \subset$

$M_1 \times \cdots \times M_n$. There are projections $p_{[1,n],i} = p_i : M_{[1,n]} \to M_i$, $\pi_i : M_1 \times \cdots \times M_n \to M_i$, and we have a commutative diagram:

$$\begin{tikzcd}
M_{[1,n]} \arrow[r, hook] \arrow[d, "p_{[1,n],I}"'] \arrow[dd, bend right=60, "p_i"'] & M_1 \times \cdots \times M_r \arrow[ddl, "\pi_i"] \\
M_I \arrow[d, "p_{I,i}"'] & \\
M_i &
\end{tikzcd}$$

Let $\{I_1, \cdots, I_r\}$ be a partition of an interval $I = [j,k]$ into sub-intervals, namely there is an increasing sequence $j = i_1 < \cdots < i_{r+1} = k+1$ such that $I_a = [i_a, i_{a+1} - 1]$. Then there are projections

$$M_{I_a} \to Y_{i_{a+1}-1} \leftarrow M_{I_{a+1}} \ .$$

So after renumbering

$$M'_a = M_{I_a}, \quad Y'_\ell = Y_{i_{\ell+1}-1}$$

we have another sequence of fiberings indexed by $[1,r]$. Thus $M_{I_1} \diamond \cdots \diamond M_{I_r}$ makes sense and coincides with M_I.

In what follows we fix a sequence of integers $a_i \in \mathbb{Z}$, and take the complexes $\mathcal{Z}_{a_i}(M_i)$. To an interval $I = [j,k]$ we assign the integer

$$a_I = \sum_{i=j}^{k} a_i - \sum_{i=j}^{k-1} \dim Y_i$$

and take it as the dimension of the cycle complex $\mathcal{Z}(M_I)$. With this agreement we will drop the dimensions from the notation.

Proposition. For a set of elements $\alpha_i \in \mathcal{Z}(M_i, m_i), i \in [1,n]$, the following conditions are equivalent:

(i) The set of cycles $\{p_i^* \alpha_i (i = 1, \cdots, n)$, faces$\}$ is properly intersecting in $M_{[1,n]} \times \square^{m_1 + \cdots m_n}$.

(ii) The set of cycles $\{\pi_i^* \alpha_i (i = 1, \cdots, n), M_{[1,n]}$, faces$\}$ is properly intersecting in $M_1 \times \cdots \times M_n \times \square^{m_1 + \cdots m_n}$.

When this condition is satisfied we will say that the set $\{\alpha_i (i = 1, \cdots, n)$, faces$\}$ is properly intersecting in $M_{[1,n]}$. The equivalence follows from the obvious

Lemma. *Let X be a smooth variety and $Y \subset X$ a smooth subvariety (both assumed to be equi-dimensional). For a set of cycles $\alpha_1, \cdots, \alpha_n$ on X, the following are equivalent.*

(i) The set $\{\alpha_1, \cdots, \alpha_n, Y\}$ is properly intersecting in X.

(ii) The set $\{\alpha_1, \cdots, \alpha_n\}$ is properly intersecting in X, the intersection $\alpha_i \cap Y$ is proper for each i, and the set $\{\alpha_i \cdot Y\}$ is properly intersecting in Y.

We define the subcomplex

$$\mathcal{Z}(M_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n) \subset \mathcal{Z}(M_1)\otimes\cdots\otimes\mathcal{Z}(M_n)$$

to be the one generated by elements $\alpha_1 \otimes \cdots \otimes \alpha_n$, with each α_i irreducible non-degenerate, and $\{\alpha_1, \cdots, \alpha_n, \text{ faces}\}$ properly intersecting in $M_{[1,n]}$. It is also denoted by $\widehat{\bigotimes}_{i\in[1,n]}\mathcal{Z}(M_i)$.

The proposition shows that it coincides with the distinguished subcomplex with respect to $W = \{M_{[1,n]}\}$ (the set consisting of one closed set):

$$\mathcal{Z}(M_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n) = [\mathcal{Z}(M_1)\otimes\cdots\otimes\mathcal{Z}(M_n)]_{\{M_{[1,n]}\}} \ .$$

Let $\alpha_i \in \mathcal{Z}(M_i, m_i), i \in [1,n]$ be elements such that $\{\alpha_1, \cdots, \alpha_n, \text{faces}\}$ is properly intersecting in $M_{[1,n]}$. Then for each interval $I = [j,k] \subset [1,n]$, the set $\{\alpha_j, \cdots, \alpha_k\}$ satisfies an analogous condition, thus

$$\alpha_j \otimes \cdots \otimes \alpha_k \in \mathcal{Z}(M_j)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_k) \ .$$

[To see this note the projection $M_{[1,n]} \to M_I$ is smooth, and the pull-back by a smooth map preserves the proper intersection property of cycles.] Thus the intersection in $M_I \times \square^{m_j+\cdots+m_k}$

$$(p^*_{I,j}\alpha_j)\cdot \quad \cdots \quad \cdot(p^*_{I,k}\alpha_k)$$

is defined and $\in \mathcal{Z}(M_I)$. This is denoted by

$$\alpha_j \underset{Y_j}{\circ} \cdots \underset{Y_{k-1}}{\circ} \alpha_k = \alpha_j \circ \cdots \circ \alpha_k$$

or just by α_I.

If $I_1, \cdots, I_r$ is a partition of I then

$$(p^*_{I,I_1}\alpha_{I_1}, \cdots, p^*_{I,I_r}\alpha_{I_r}) \in \mathcal{Z}(M_{I_1})\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_{I_r})$$

and

$$\alpha_{I_1} \circ \cdots \circ \alpha_{I_r} = \alpha_I$$

in $\mathcal{Z}(M_I)$. So one has the product map

$$\rho(I_1, \cdots, I_r) : \mathcal{Z}(M_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n) \to \mathcal{Z}(M_{I_1})\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_{I_r}) \ ,$$

that maps $(\alpha_1, \cdots, \alpha_n)$ to $(\alpha_{I_1}, \cdots, \alpha_{I_r})$.

(1.8) *Properties of the product map $\rho(I_1, \cdots, I_r)$.*

(0) One clearly has, for any partition $I_1, \cdots, I_r$ of $[1,n]$, the inclusion

$$\widehat{\bigotimes_{[1,n]}}\mathcal{Z}(M_i) \subset (\widehat{\bigotimes_{I_1}}\mathcal{Z}(M_i)\,)\otimes\cdots\otimes(\widehat{\bigotimes_{I_r}}\mathcal{Z}(M_i)\,) \ .$$

The product map $\rho(I_1, \cdots, I_r)$ satisfies the following properties.

(1) If $K_1, \cdots, K_s$ is a partition of $[1,r]$, namely $K_b = [k_b, k_{b+1}-1]$ with $k_1 = 1, k_{s+1}-1 = r$, let

$$J_b = I_{k_b} \cup \cdots \cup I_{k_{b+1}-1} \ , \quad 1 \le b \le s \ .$$

Then $J_1, \cdots, J_s$ is another partition of $[1,n]$. Let $M'_a = M_{I_a}$ and $Y'_\ell = Y_{i_{\ell+1}-1}$. Then $M'_{K_b} = M_{J_b}$. We have the product map $\rho(K_1, \cdots, K_s) : \widehat{\bigotimes}_{[1,r]} \mathcal{Z}(M'_a) \to \widehat{\bigotimes}_{[1,s]} \mathcal{Z}(M'_{K_b})$, namely $\rho(K_1, \cdots, K_s) : \widehat{\bigotimes}_{[1,r]} \mathcal{Z}(M_{I_a}) \to \widehat{\bigotimes}_{[1,s]} \mathcal{Z}(M_{J_b})$. The following diagram commutes:

$$\begin{tikzcd}
\widehat{\bigotimes}_{[1,n]} \mathcal{Z}(M_i) \arrow[rr, "\rho(I_1, \cdots, I_r)"] \arrow[dr, "\rho(J_1, \cdots, J_s)"'] & & \widehat{\bigotimes}_{[1,r]} \mathcal{Z}(M_{I_a}) \arrow[dl, "\rho(K_1, \cdots, K_s)"] \\
& \widehat{\bigotimes}_{[1,s]} \mathcal{Z}(M_{J_b}) &
\end{tikzcd}$$

(2) The following square commutes (where the vertical maps are inclusions):

$$\begin{tikzcd}
\widehat{\bigotimes}_{[1,n]} \mathcal{Z}(M_i) \arrow[r, "\rho(I_1, \cdots, I_r)"] \arrow[d] & \widehat{\bigotimes}_{[1,r]} \mathcal{Z}(M_{I_a}) \arrow[d] \\
\widehat{\bigotimes}_{I_1} \mathcal{Z}(M_i) \otimes \cdots \otimes \widehat{\bigotimes}_{I_r} \mathcal{Z}(M_i) \arrow[r, "\rho(I_1) \otimes \cdots \otimes \rho(I_r)"] & \bigotimes_{[1,n]} \mathcal{Z}(M_{I_a}) \ .
\end{tikzcd}$$

Remark. For the map $\rho(I_1, \cdots, I_r)$ the following labeling will be useful as well (which we have employed before). To a partition $I_a = [i_a, i_{a+1}-1]$ as above, one can associate a subset $S \subset [1, n-1]$ given by

$$S = \coprod_a [i_a, i_{a+1} - 2]$$

(remove from each I_a the terminal element, and take disjoint union for a). Note the varieties M_{I_a} are obtained by taking fiber products over Y_i with $i \in S$. The sequence of dimensions for the target of $\rho(I_1, \cdots, I_r)$ is

$$\sum_{i \in I_a} s_i - \sum_{i \in S \cap I_a} \dim Y_i \ , \quad i = 1, \cdots, r.$$

Giving a partition to sub-intervals is equivalent to giving a subset S of $[1, n-1]$. One may write ρ_S in place of $\rho(I_1, \cdots, I_r)$. Then the commutativity (1) can be written:

$$\rho_S = \rho_{S''} \rho_{S'}$$

whenever S is the disjoint union of S' and S''. (Let S' correspond to the partition $I_1, \cdots, I_r$, $S'' \subset [1, r-1]$ correspond to $K_1, \cdots, K_s$, and S correspond to $J_1, \cdots, J_s$. By means of the renumbering $\varphi : [1, r-1] \to [1, n-1]$ given by $\varphi(\ell) = i_{\ell+1} - 1$, S'' can be identified with a subset of $[1, n-1]$. Then one has $S = S' \amalg S''$.)

(1.9) *Properties of* $\widehat{\bigotimes}\mathcal{Z}(M_i,\mathcal{U}_i)$. This subsection is a continuation of (1.5). We list the properties of the complex $\mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n)$.

(a) The subcomplex is functorial in $\mathcal{U}_i$. If $\mathcal{U}_i \to \mathcal{V}_i$ are maps in $Cov(M_i)$, there is an induced map

$$\mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) \to \mathcal{Z}(M_1,\mathcal{V}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{V}_n)\ .$$

(b) The map $\rho(I_1,\cdots,I_r)$ is functorial in $\mathcal{U}_i$. If $\mathcal{U}_i \to \mathcal{V}_i$ are maps of coverings the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) & \xrightarrow{\rho(I_1,\cdots,I_r)} & \mathcal{Z}(M_{I_1},\mathcal{U}_{I_1})\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_{I_r},\mathcal{U}_{I_r}) \\
\downarrow & & \downarrow \\
\mathcal{Z}(M_1,\mathcal{V}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{V}_n) & \xrightarrow{\rho(I_1,\cdots,I_r)} & \mathcal{Z}(M_{I_1},\mathcal{V}_{I_1})\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_{I_r},\mathcal{V}_{I_r})\ .
\end{array}$$

Among the following properties, (0)-(2) are parallel to those in the previous subsection.

(0) For any partition $I_1,\cdots,I_r$ of $[1,n]$, there is inclusion

$$\widehat{\bigotimes_{[1,n]}}\mathcal{Z}(M_i,\mathcal{U}_i) \subset (\widehat{\bigotimes_{I_1}}\mathcal{Z}(M_i,\mathcal{U}_i)\,)\otimes\cdots\otimes(\widehat{\bigotimes_{I_r}}\mathcal{Z}(M_i,\mathcal{U}_i)\,)\ .$$

(1) If $K_1,\cdots,K_s$ is a partition of $[1,r]$, and $J_1,\cdots,J_s$ is the resulting partition of $[1,n]$, then $\rho(K_1,\cdots,K_s)\rho(I_1,\cdots,I_r)=\rho(J_1,\cdots,J_s)$.

(2) The following square commutes (where the vertical maps are inclusions):

$$\begin{array}{ccc}
\widehat{\bigotimes}_{[1,n]}\mathcal{Z}(M_i,\mathcal{U}_i) & \xrightarrow{\rho(I_1,\cdots,I_r)} & \widehat{\bigotimes}_{[1,r]}\mathcal{Z}(M_{I_a},\mathcal{U}_{I_a}) \\
\downarrow & & \downarrow \\
\widehat{\bigotimes}_{I_1}\mathcal{Z}(M_i,\mathcal{U}_i)\otimes\cdots\otimes\widehat{\bigotimes}_{I_r}\mathcal{Z}(M_i,\mathcal{U}_i) & \xrightarrow{\rho(I_1)\otimes\cdots\otimes\rho(I_r)} & \bigotimes_{[1,n]}\mathcal{Z}(M_{I_a,}\mathcal{U}_{I_a})\ .
\end{array}$$

(3) If $\pi : N_1 \to M_1$ is a smooth map, there is the corresponding map

$$\pi^*\otimes id : \mathcal{Z}(M_1,\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) \to \mathcal{Z}(N_1,\pi^{-1}\mathcal{U}_1)\hat{\otimes}\mathcal{Z}(M_2,\mathcal{U}_2)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n)\ .$$

This is functorial in $\mathcal{U}_i$. The same in each M_i. If $\pi : M_1 \to N_1$ is a projective map, there is the map, functorial in $\mathcal{U}_i$,

$$\pi_*\otimes id : \mathcal{Z}(M_1,\pi^{-1}\mathcal{U}_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n) \to \mathcal{Z}(N_1,\mathcal{U}_1)\hat{\otimes}\mathcal{Z}(M_2,\mathcal{U}_2)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n,\mathcal{U}_n)\ .$$

The quasi-isomorphisms $\iota : \mathcal{Z}(A_i) \to \mathcal{Z}(M_i,\mathcal{U}_i)$ induce a quasi-isomorphism

$$\otimes\iota : \bigotimes_{[1,n]}\mathcal{Z}(A_i) \to \bigotimes_{[1,n]}\mathcal{Z}(M_i,\mathcal{U}_i)\ .$$

Composing with the inverse of the inclusion $\widehat{\bigotimes}\mathcal{Z}(M_i,\mathcal{U}_i) \hookrightarrow \bigotimes\mathcal{Z}(M_i,\mathcal{U}_i)$, one obtains an isomorphism in the *derived category*

$$\iota : \bigotimes_{[1,n]}\mathcal{Z}(A_i) \to \widehat{\bigotimes_{[1,n]}}\mathcal{Z}(M_i,\mathcal{U}_i)\ .$$

(It is a slight abuse of notation to use the same ι for a map in the derived category.) For a partition $I_1,\cdots,I_r$ of $[1,n]$, there is a unique map in the derived category

$$\rho(I_1,\cdots,I_r) : \mathcal{Z}(A_1)\otimes\cdots\otimes\mathcal{Z}(A_n) \to \mathcal{Z}(A_{I_1})\otimes\cdots\otimes\mathcal{Z}(A_{I_r})$$

which makes the following diagram commute:

$$\begin{array}{ccc}
\widehat{\bigotimes}_{[1,n]} \mathcal{Z}(M_i, \mathcal{U}_i) & \xrightarrow{\rho(I_1,\cdots,I_r)} & \widehat{\bigotimes}_{[1,r]} \mathcal{Z}(M_{I_t}, \mathcal{U}_{I_t}) \\
{\scriptstyle\iota}\uparrow & & \uparrow{\scriptstyle\iota} \\
\bigotimes_{[1,n]} \mathcal{Z}(A_i) & \xrightarrow{\rho(I_1,\cdots,I_r)} & \bigotimes_{[1,r]} \mathcal{Z}(A_{I_t}) \ .
\end{array}$$

The maps ι and ρ satisfy the following properties.

(4) For $\rho(I_1, \cdots, I_r)$ one has commutativity analogous to (1) above.

(5) If $\pi : N_1 \to M_1$ is a smooth map, the map

$$\pi^* \otimes id : \mathcal{Z}(A_1) \otimes \mathcal{Z}(A_2) \otimes \cdots \otimes \mathcal{Z}(A_n) \to \mathcal{Z}(\pi^{-1}A_1) \otimes \mathcal{Z}(A_2) \otimes \cdots \otimes \mathcal{Z}(A_n)$$

and the $\pi^* \otimes id$ in (3) above are compatible via the maps ι. If π is a projective map

$$\pi_* \otimes id : \mathcal{Z}(\pi^{-1}A_1) \otimes \mathcal{Z}(A_2) \otimes \cdots \otimes \mathcal{Z}(A_n) \to \mathcal{Z}(A_1) \otimes \mathcal{Z}(A_2) \otimes \cdots \otimes \mathcal{Z}(A_n)$$

and the $\pi_* \otimes id$ in (3) above are compatible via the maps ι.

2 Function complexes $F(X_1, \cdots, X_n)$

(2.1) For an integer $n \geq 2$, let $[1, n] = \{1, \ldots, n\}$. We will consider subsets I of $[1, n]$ with cardinality ≥ 2. Such I is a finite ordered set. For notions regarding finite ordered sets, see (0.5). In particular recall for a subset $\Sigma \subset \overset{\circ}{I}$ there corresponds a segmentation.

In what follows we will consider sequences of varieties parametrized by I. It is often convenient to give definitions and constructions in case $I = [1, n]$.

(2.2) Let S be a quasi-projective variety and $X_1, \cdots, X_n$ be smooth quasi-projective varieties, each equipped with a projective map to S (we call such X_i a sequence of varieties over S). For a subset $I \subset [1, n]$, let $X_I = \prod_{i \in I} X_i$ (product over k). So $X_{[1,n]} = X_1 \times \cdots \times X_n$.

For a non-empty subset $I \subset [1, n]$, let $X_{[1,n]} \overset{\sim}{\to} \prod\limits_{i \in I} X_i \times \prod\limits_{i \notin I} X_i$ be the natural isomorphism (switching factors); define the closed subset $A_I \subset X_{[1,n]}$ by the Cartesian square

$$\begin{array}{ccc}
X_{[1,n]} & \xrightarrow{\sim} & \prod\limits_{i \in I} X_i \times \prod\limits_{i \notin I} X_i \\
\uparrow & & \uparrow \\
A_I & \xrightarrow{\sim} & \prod\limits_{i \in I}{}_S X_i \times \prod\limits_{i \notin I} X_i
\end{array}$$

where $\prod_S$ denotes fiber product over S.

- For example, if I consists of a single element, $A_I = X_{[1,n]}$; if $I = \{1, 2\}$, $A_I = (X_1 \times_S X_2) \times X_3 \times \cdots \times X_n$; if $I = [1, n]$, $A_{[1,n]} = X_1 \times_S X_2 \times_S \cdots \times_S X_n$.
- If $I \subset I'$, then $A_I \supset A_{I'}$. For two subsets I and I' with non-empty intersection, $A_{I \cup I'} = A_I \cap A_{I'}$.

Let $U_I = X_{[1,n]} - A_I$. $U_{[1,n]}$ is the complement of $X_1 \times_S X_2 \times_S \cdots \times_S X_n$. If $I \subset I'$, then $U_I \subset U_{I'}$. If $I \cap I'$ is non-empty, $U_{I \cup I'} = U_I \cup U_{I'}$.

Let $\mathcal{J}$ be a subset of $(1, n) = [2, n-1]$. If $\mathcal{J} = \{j_1 \cdots j_r\}$, recall the associated intervals are given by $J^k = [j_k, j_{k+1}]$ for $k = 0, \cdots, r$ with $j_0 = 1$ and $j_{r+1} = n$. To each J^k there corresponds the closed set $A_{J^k} \subset X_{[1,n]} = X_1 \times \cdots \times X_n$ and its complement U_{J^k}. The intersection of A_{J^k}'s is $A_{[1,n]}$, and the union of U_{J^k}'s is $U_{[1,n]}$. We thus have a covering of $U_{[1,n]}$ indexed by $[0, r]$:

$$\mathcal{U}(\mathcal{J}) = \{U_{J^0}, U_{J^1}, \cdots, U_{J^r}\} .$$

Taking $M = X_{[1,n]}$ and $\mathcal{U} = \mathcal{U}(\mathcal{J})$ in the construction of the previous section, one obtains the complex

$$\mathcal{Z}_s(X_{[1,n]}, \mathcal{U}(\mathcal{J})) .$$

As before the differential is denoted d, and when necessary we write $\mathcal{Z}_s(X_{[1,n]}, \mathcal{U}(\mathcal{J}))^\bullet$ where the upper indexing is the cohomological degree. There is a natural quasi-isomorphism

$$\iota : \mathcal{Z}_s(X_1 \times_S \times \cdots \times_S X_n) \to \mathcal{Z}_s(X_{[1,n]}, \mathcal{U}(\mathcal{J})) .$$

Note in the discussion so far, one can replace $[1, n]$ by any subset $\mathbb{I}$ of $[1, n]$ and a subset $\mathcal{J} \subset \mathring{\mathbb{I}}$. More specifically, we have:

- One has the product

$$X_{\mathbb{I}} = \prod_{i \in \mathbb{I}} X_i .$$

Associated to a subset $I \subset \mathbb{I}$ is a closed set $A_I \subset X_{\mathbb{I}}$ and its complement U_I (to be specific, we write $A_{I \subset \mathbb{I}}$ and $U_{I \subset \mathbb{I}}$). In particular, $A_{\mathbb{I}}$ is the fiber product of all X_i over S, and $U_{\mathbb{I}}$ its complement.

- For a set $\mathcal{J} \subset \mathring{\mathbb{I}}$ of cardinality r, there corresponds a set of intervals

$$J^i = J^i(\mathcal{J} \subset \mathbb{I}) , \quad 0 \le i \le r ,$$

of $\mathbb{I}$. Thus we have an $[0, r]$-covering of $U_{\mathbb{I}}$

$$\mathcal{U}(\mathcal{J}) = \mathcal{U}(\mathcal{J} \subset \mathbb{I}) = \{U_{J^0}, \cdots, U_{J^r}\} .$$

This gives us the a complex $\mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J} \subset \mathbb{I}))$ equipped with a quasi-isomorphism from $\mathcal{Z}(A_{\mathbb{I}})$.

- Note $U_{I \subset \mathbb{I}}$ is an open set of $X_{\mathbb{I}}$; it should be distinguished from the open set $U_{I \subset [1,n]} \subset X_{[1,n]}$.

We have natural maps between such complexes, the restriction and the projection.

(1) For $\mathcal{J} \subset \mathcal{J}'$, one has the *restriction map*, which is a quasi-isomorphism:

$$\mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J})) \to \mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J}')) .$$

To define it, assume $\mathbb{I} = [1, n]$ for simplicity. Let $\mathcal{J} = \{j_1, \cdots, j_r\}$, $J^k = [j_k, j_{k+1}]$ for $0 \le k \le r$ as above. Let $\mathcal{J}' = \{j'_1, \cdots, j'_{r'}\}$, and define a map $\lambda : [0, r'] \to [0, r]$ as follows. For each j'_t there is a unique k such that $j_{k-1} < j'_t \le j_k$. Then $J'_t = [j'_{t-1}, j'_t] \subset [j_{k-1}, j_k] = J_k$. If we set $\lambda(t) = k$, then λ is order-preserving and $U_{J'_t} \subset U_{J_k}$; in other words $\lambda : \mathcal{U}(\mathcal{J}) \to \mathcal{U}(\mathcal{J}')$ is a map of coverings. It induces the map between the Čech cycle complexes as stated.

(2) *Assume now that S is projective.* For $\ell \in \mathring{\mathbb{I}} - \mathcal{J}$, we have the *projection* along X_ℓ

$$\pi_{X_\ell} : \mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J} \subset \mathbb{I})) \to \mathcal{Z}(X_{\mathbb{I}-\{\ell\}}, \mathcal{U}(\mathcal{J} \subset \mathbb{I} - \{\ell\})) \ .$$

The definition in case $\mathbb{I} = [1, n]$ is as follows. Let $p : X_{[1,n]} \to X_{[1,n]-\{\ell\}}$ be the projection. One has $p^{-1}(U_I) = U_I$ for a subset $I \subset [1, n] - \{\ell\}$ (more precisely, $p^{-1}(U_{I\subset[1,n]-\{\ell\}}) = U_{I\subset[1,n]}$.)

If $\ell \in (j_k, j_{k+1})$, the associated intervals to $\mathcal{J} \subset [1, n] - \{\ell\}$ are

$$\{\bar{J}^i\} = \{J^0, \cdots, J^{k-1}, J^k - \{\ell\}, J^{k+1}, \cdots, J^r\} \ ,$$

and the associated open covering is $\{U_{\bar{J}^i}\}$. Since

$$p^{-1}U_{\bar{J}^i} = U_{\bar{J}^i} \subset U_{J^i} \ ,$$

one has the restriction map

$$\mathcal{Z}(X_{[1,n]}, \{U_{J^0}, \cdots, U_{J^r}\}) \to \mathcal{Z}(X_{[1,n]}, \{U_{\bar{J}^0}, \cdots, U_{\bar{J}^r}\}) \ .$$

Composing it with the projection

$$p_* : \mathcal{Z}(X_{[1,n]}, \{U_{\bar{J}^0}, \cdots, U_{\bar{J}^r}\}) \to \mathcal{Z}(X_{[1,n]-\{\ell\}}, \{U_{\bar{J}^0}, \cdots, U_{\bar{J}^r}\})$$

one obtains the stated map.

More generally for a subset $K \subset \mathring{\mathbb{I}} - \mathcal{J}$ one has the corresponding projection

$$\pi_K : \mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J} \subset \mathbb{I})) \to \mathcal{Z}(X_{\mathbb{I}-K}, \mathcal{U}(\mathcal{J} \subset \mathbb{I} - K)) \ .$$

If $K = K' \cup K''$, $\pi_K = \pi_{K''}\pi_{K'}$, namely the following diagram commutes.

$$\begin{tikzcd}
\mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J})) \arrow[rr, "\pi_K"] \arrow[dr, "\pi_{K'}"'] & & \mathcal{Z}(X_{\mathbb{I}-K}, \mathcal{U}(\mathcal{J})) \\
& \mathcal{Z}(X_{\mathbb{I}-K'}, \mathcal{U}(\mathcal{J})) \arrow[ur, "\pi_{K''}"'] &
\end{tikzcd}$$

In particular, $\pi_{K''}\pi_{K'} = \pi_{K'}\pi_{K''}$.

(3) The quasi-isomorphism $\iota : \mathcal{Z}(A_{\mathbb{I}}) \to \mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J}))$ is compatible with restriction maps and projections. It means, for projection, the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J})) \arrow[r, "\pi_K"] & \mathcal{Z}(X_{\mathbb{I}-K}, \mathcal{U}(\mathcal{J})) \\
\mathcal{Z}(A_{\mathbb{I}}) \arrow[u, "\iota"] \arrow[r, "\pi_K"] & \mathcal{Z}(A_{\mathbb{I}-K}) \arrow[u, "\iota"']
\end{tikzcd} \ .$$

Here π_K at the bottom is the map induced by the projection $A_{\mathbb{I}} \to A_{\mathbb{I}-K}$.

We would like to have projection maps as above in general, under the assumption S quasi-projective and $X_i \to S$ projective.

Let $S \hookrightarrow \bar{S}$ a compactification, namely an open immersion to a projective variety. For each X_i take a projective variety $\bar{X}_i$ with a projective map $\bar{X}_i \to \bar{S}$ extending p_i. (We say $\bar{X}_i/\bar{S}$ is a compactification of X_i/S.)

Then one has

$$X_{[1,n]} = \prod X_i \qquad \text{and} \qquad \bar{X}_{[1,n]} := \prod \bar{X}_i \ .$$

To $I \subset [1,n]$, there corresponds a closed set $A_I \subset X_{[1,n]}$ and its complement U_I (resp. $\bar{A}_I \subset \bar{X}_{[1,n]}$ and its complement $\bar{U}_I$).

Given $\mathcal{J} \subset (1,n)$, we define a partial compactification by

$$X^{\mathcal{J}}_{[1,n]} := \prod_{i \in [1,n]} X'_i \quad \text{with} \quad X'_i = \begin{cases} \bar{X}_i & \text{if } i \in (1,n) - \mathcal{J} \\ X_i & \text{if } i \in \{1,n\} \cup \mathcal{J} \end{cases}$$

For $I \subset [1,n]$, define the closed subset $A^{\mathcal{J}}_I \subset X^{\mathcal{J}}_{[1,n]}$ by the following diagram:

$$\begin{array}{ccc} X^{\mathcal{J}}_{[1,n]} & \xrightarrow{\sim} & \prod_{i \in I} X'_i \times \prod_{i \notin I} X'_i \\ \uparrow & & \uparrow \\ A^{\mathcal{J}}_I & \xrightarrow{\sim} & \prod_{i \in I} {}_S X'_i \times \prod_{i \notin I} X'_i \ . \end{array}$$

If $I \subset I'$ then $A^{\mathcal{J}}_I \supset A^{\mathcal{J}}_{I'}$. For two subsets I and I' with non-empty intersection, $A^{\mathcal{J}}_{I \cup I'} = A^{\mathcal{J}}_I \cap A^{\mathcal{J}}_{I'}$. Further, if $I \cap (\{1,n\} \cup \mathcal{J}) \neq \emptyset$, namely if $X'_i = X_i$ for some element $i \in I$, then $A^{\mathcal{J}}_I = A_I$. In particular, $A^{\mathcal{J}}_{[1,n]} = A_{[1,n]} = X_1 \times_S \cdots \times_S X_n$.

Let $U^{\mathcal{J}}_I = X^{\mathcal{J}}_{[1,n]} - A^{\mathcal{J}}_I$. If $I \subset I'$ then $U^{\mathcal{J}}_I \subset U^{\mathcal{J}}_{I'}$; if $I \cap I' \neq \emptyset$, $U^{\mathcal{J}}_{I \cup I'} = U^{\mathcal{J}}_I \cup U^{\mathcal{J}}_{I'}$. Note $U^{\mathcal{J}}_{[1,n]} = U_{[1,n]}$.

$$\begin{array}{ccc} X_{[1,n]} & \subset & X^{\mathcal{J}}_{[1,n]} \\ \cup & & \cup \\ U_I & \subset & U^{\mathcal{J}}_I \end{array}$$

The $\mathcal{J}$ specifies $\mathcal{U}(\mathcal{J})$, a covering of $U^{\mathcal{J}}_{[1,n]} = U_{[1,n]}$. So we have the complex $\mathcal{Z}(X^{\mathcal{J}}_{[1,n]}, \mathcal{U}(\mathcal{J}))$ and a quasi-isomorphism $\mathcal{Z}(X_1 \times_S \times \cdots \times_S X_n) \to \mathcal{Z}(X^{\mathcal{J}}_{[1,n]}, \mathcal{U}(\mathcal{J}))$.

As before the same construction can be applied to a subset $\mathbb{I}$ of $[1,n]$, and a subset $\mathcal{J} \subset \overset{\circ}{\mathbb{I}}$. One has the product $X_{\mathbb{I}}$ and its partial compactification $X^{\mathcal{J}}_{\mathbb{I}}$. Each subset $I \subset \mathbb{I}$ corresponds to a closed set $A^{\mathcal{J}}_I$. A subset $\mathcal{J} \subset \overset{\circ}{\mathbb{I}}$ gives a covering $\mathcal{U}(\mathcal{J})$ of $U_{\mathbb{I}}$, and thus the complex $\mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}}, \mathcal{U}(\mathcal{J}))$ quasi-isomorphic to $\mathcal{Z}(A_{\mathbb{I}})$.

We have again the following maps with similar properties.

(1) For $\mathcal{J} \subset \mathcal{J}'$, we have $X^{\mathcal{J}}_{\mathbb{I}} \supset X^{\mathcal{J}'}_{\mathbb{I}}$ and $\mathcal{U}(\mathcal{J} \subset \mathbb{I}) \supset \mathcal{U}(\mathcal{J}' \subset \mathbb{I})$. Hence the restriction map (which is a quasi-isomorphism)

$$\mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}}, \mathcal{U}(\mathcal{J})) \to \mathcal{Z}(X^{\mathcal{J}'}_{\mathbb{I}}, \mathcal{U}(\mathcal{J}')) \ .$$

(2) For $\ell \in \mathbb{I} - \mathcal{J}$, not containing either end of $\mathbb{I}$,

$$\pi_{X_\ell} : \mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}}, \mathcal{U}(\mathcal{J})) \to \mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I} - \{\ell\}}, \mathcal{U}(\mathcal{J})) \ .$$

This is defined in the same way as before, since the projection $p : X^{\mathcal{J}}_{\mathbb{I}} = X^{\mathcal{J}}_{\mathbb{I} - \{\ell\}} \times \bar{X}_\ell \to X^{\mathcal{J}}_{\mathbb{I} - \{\ell\}}$ is projective.

More generally for $K \subset \overset{\circ}{\mathbb{I}} - \mathcal{J}$ one has the projection $\pi_K : \mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}}, \mathcal{U}(\mathcal{J} \subset \mathbb{I})) \to \mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}-K}, \mathcal{U}(\mathcal{J} \subset \mathbb{I} - K))$. If $K = K' \cup K''$, $\pi_K = \pi_{K''}\pi_{K'}$.

(3) The quasi-isomorphism $\iota : \mathcal{Z}(A_{\mathbb{I}}) \to \mathcal{Z}(X^{\mathcal{J}}_{\mathbb{I}}, \mathcal{U}(\mathcal{J}))$ is compatible with restrictions and projections.

(2.3) *The complex* $\mathcal{F}(I)$. For simplicity let

$$\mathcal{F}([1,n], \mathcal{J}) = \mathcal{Z}(X^{\mathcal{J}}_{[1,n]}, \mathcal{U}(\mathcal{J})) \ ;$$

the differential is denoted d , and write $\mathcal{F}([1,n], \mathcal{J})^{\bullet}$ to specify the grading.

There is the restriction map, for $\mathcal{J} \subset \mathcal{J}'$ with $|\mathcal{J}'| = |\mathcal{J}| + 1$,

$$r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}([1,n], \mathcal{J}) \to \mathcal{F}([1,n], \mathcal{J}') \ .$$

This is a quasi-isomorphism. If $\mathcal{J}' = \mathcal{J} \cup \{k\}$, let $\mathcal{J}_{>k} = \{i \in \mathcal{J} | i > k\}$, and $|\mathcal{J}_{>k}|$ its cardinality; define the map

$$r : \mathcal{F}([1,n], \mathcal{J}) \to \mathcal{F}([1,n], \mathcal{J}')$$

to be $(-1)^{|\mathcal{J}_{>k}|} r_{\mathcal{J},\mathcal{J}'}$.

Let

$$A^{a,p} = \bigoplus_{a=|\mathcal{J}|+1} \mathcal{F}([1,n], \mathcal{J})^p$$

the sum over $\mathcal{J}$ with $a = |\mathcal{J}| + 1$. Then one has $rr = 0$. With differentials r, d, this forms a “double” complex. The total complex $\mathrm{Tot}(A)$ is a complex with differential $r + (-1)^a d$ (which will be also be denoted by d if no confusion is likely) on $A^{a,p}$. Define

$$\mathcal{F}([1,n]) = \mathrm{Tot}(A) \ .$$

The same construction applies to any finite subset $I \subset [1,n]$, so that one has a complex $\mathcal{F}(I)$. It has complexes $\mathcal{F}(I, \mathcal{J})[-(|\mathcal{J}| + 1)]$ as subquotients. Here recall for a complex $(K^{\bullet}, d)$, the shift $K^{\bullet}[1]$ is defined by $(K^{\bullet}[1])^p = K^{p+1}$ and $d_{K[1]} = -d$.

If $|I| = 2$, $\mathcal{F}(I) = \mathcal{F}(I, \emptyset)[-1]$, so there is a quasi-isomorphism $\mathcal{Z}(A_I)[-1] \to \mathcal{F}(I)$. If $|I| \geq 3$, $\mathcal{F}(I)$ is acyclic. This follows from the lemma below.

(2.4) Let T be a non-empty finite ordered set, and $\mathcal{P}(T)$ be the set of subsets $S \subset T$ (including the empty set). Suppose to each $S \in \mathcal{P}(T)$ there corresponds a complex $C_S \in C(Ab)$, and to each inclusion $S \subset S'$ there corresponds a map of complexes $f_{S\,S'} : C_S \to C_{S'}$, satisfying $f_{S\,S} = id$, and $f_{S'\,S''} f_{S\,S'} = f_{S\,S''}$ for $S \subset S' \subset S''$.

We then have a “double” complex

$$0 \to C_{\emptyset} \to \bigoplus_{|S|=1} C_S \to \bigoplus_{|S|=2} C_S \to \cdots \to C_T \to 0 \ .$$

Here the maps are signed sums of the maps $f_{S\,S'}$ with $|S'| = |S| + 1$, the signs being specified as in (2.3). We can form its total complex $\mathrm{Tot}(C_S)$. One proves the following lemma by induction on n.

Lemma. *Assume for each $S \subset S'$ the map $f_{S\,S'}$ is a quasi-isomorphism. Then* $\mathrm{Tot}(C_S)$ *is acyclic.*

(2.5) *The complex* $\mathcal{F}(I|\Sigma)$ *and the product map* ρ. Let $\Sigma \subset \mathring{I}$ and $I_1, \cdots, I_c$ be the segmentation of I by Σ. Set

$$\mathcal{F}(I \top \Sigma) := \mathcal{F}(I_1) \otimes \mathcal{F}(I_2) \otimes \cdots \otimes \mathcal{F}(I_c) \ ,$$

the tensor product of the simple complexes $\mathcal{F}(I_i)$. The differential, also denoted d, is given by

$$d(\alpha_1 \otimes \cdots \otimes \alpha_c) = \sum (-1)^{\sum_{j>i} \deg \alpha_j} \alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes d(\alpha_i) \otimes \cdots \otimes \alpha_c \ .$$

As a module $\mathcal{F}(I \top \Sigma)$ is the direct sum of $\mathcal{F}(I_1, \mathcal{J}_1) \otimes \mathcal{F}(I_2, \mathcal{J}_2) \otimes \cdots \otimes \mathcal{F}(I_c, \mathcal{J}_c)$, where $\mathcal{J}_i$ varies over subsets of $\mathring{I}_i$.

One has quasi-isomorphic c-tuple subcomplexes

$$\mathcal{F}(I_1, \mathcal{J}_1) \hat{\otimes} \mathcal{F}(I_2, \mathcal{J}_2) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c, \mathcal{J}_c) \subset \mathcal{F}(I_1, \mathcal{J}_1) \otimes \mathcal{F}(I_2, \mathcal{J}_2) \otimes \cdots \otimes \mathcal{F}(I_c, \mathcal{J}_c) \ .$$

The sum of them

$$\mathcal{F}(I|\Sigma) = \mathcal{F}(I_1) \hat{\otimes} \mathcal{F}(I_2) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c) := \bigoplus \mathcal{F}(I_1, \mathcal{J}_1) \hat{\otimes} \mathcal{F}(I_2, \mathcal{J}_2) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c, \mathcal{J}_c)$$

is a subcomplex of $\mathcal{F}(I \top \Sigma)$ if the differential d is defined by the same formula as above, and the inclusion $\mathcal{F}(I|\Sigma) \to \mathcal{F}(I \top \Sigma)$ is a quasi-isomorphism.

For our convenience we write

$$\mathcal{F}(I, \mathcal{J} \top \Sigma) = \mathcal{F}(I_1, \mathcal{J}_1) \otimes \mathcal{F}(I_2, \mathcal{J}_2) \otimes \cdots \otimes \mathcal{F}(I_c, \mathcal{J}_c) \ ,$$

if $\mathcal{J} \subset \mathring{I} - \Sigma$, $I_1, \cdots, I_c$ is the segmentation of I by Σ, and $\mathcal{J}_i = \mathcal{J} \cap \mathring{I}_i$. Similarly

$$\mathcal{F}(I, \mathcal{J}|\Sigma) := \mathcal{F}(I_1, \mathcal{J}_1) \hat{\otimes} \mathcal{F}(I_2, \mathcal{J}_2) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c, \mathcal{J}_c) \ .$$

For $\mathcal{J} \subset \mathcal{J}'$ there is the corresponding map such as $r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I, \mathcal{J}'|\Sigma)$.

Recall $\mathcal{F}(I)$ was defined to be the total complex of a "double" complex. Likewise there is a "double" complex whose total complex is canonically isomorphic to $\mathcal{F}(I|\Sigma)$. We explain this in the case $|\Sigma| = 1$.

Let $A^{\bullet,\bullet}$, $B^{\bullet,\bullet}$ be the "double" complexes as above defining $\mathcal{F}([1,m])$, $\mathcal{F}([m,n])$, respectively. In $A^{a,p} \otimes B^{b,q} = \bigoplus \mathcal{F}([1,m], \mathcal{J})^p \otimes \mathcal{F}([m,n], \mathcal{J}')^q$ there is a quasi-isomorphic subcomplex

$$A^{a,p} \hat{\otimes} B^{b,q} = \bigoplus \mathcal{F}([1,m], \mathcal{J})^p \hat{\otimes} \mathcal{F}([m,n], \mathcal{J}')^q \ .$$

Recall from (0.3) that $A^{\bullet,\bullet} \times B^{\bullet,\bullet}$ is the "double" complex $E^{\bullet,\bullet}$ defined by

$$E^{c,r} = \bigoplus_{a+b=c\,,p+q=r} A^{a,p} \otimes B^{b,q}$$

and appropriate differentials d, r. Let $A^{\bullet,\bullet} \hat{\times} B^{\bullet,\bullet}$ be the "double" subcomplex given by $E_1^{c,r} = \bigoplus_{a+b=c\,,p+q=r} A^{a,p} \hat{\otimes} B^{b,q}$. By (0.3) there is an isomorphism of complexes

$$u : \mathrm{Tot}(A) \hat{\otimes} \mathrm{Tot}(B) \xrightarrow{\sim} \mathrm{Tot}(A^{\bullet,\bullet} \hat{\times} B^{\bullet,\bullet}) \ .$$

Although the groups on the two sides are identical, the differentials are different, and u is not the identity.

More generally, let $A_1^{\bullet,\bullet}, \cdots, A_c^{\bullet,\bullet}$ be the "double" complexes for $\mathcal{F}(I_1), \cdots, \mathcal{F}(I_c)$, respectively. Then there is a quasi-isomorphic "double" subcomplex $A_1^{\bullet,\bullet} \hat{\times} \cdots \hat{\times} A_c^{\bullet,\bullet}$ of $A_1^{\bullet,\bullet} \times \cdots \times A_c^{\bullet,\bullet}$ and a quasi-isomorphism of complexes

$$u : \mathcal{F}(I|\Sigma) \xrightarrow{\sim} \mathrm{Tot}(A_1^{\bullet,\bullet} \hat{\times} \cdots \hat{\times} A_c^{\bullet,\bullet}) \ .$$

We next define a map of complexes (called the product map) $\rho : \mathcal{F}([1,m]) \hat{\otimes} \mathcal{F}([m,n]) \to \mathcal{F}([1,n])$. Let $A^{\bullet,\bullet}$, $B^{\bullet,\bullet}$, $C^{\bullet,\bullet}$ be the "double" complexes as above defining $\mathcal{F}([1,m])$, $\mathcal{F}([m,n])$ and $\mathcal{F}([1,n])$, respectively. The product maps $\rho : \mathcal{F}([1,m], \mathcal{J}) \hat{\otimes} \mathcal{F}([m,n], \mathcal{J}') \to \mathcal{F}([1,n], \mathcal{J} \cup \{m\} \cup \mathcal{J}')$ define a map of "double" complexes

$$\rho : A^{\bullet,\bullet} \hat{\times} B^{\bullet,\bullet} \to C^{\bullet,\bullet} \ .$$

(One can verify the compatibility of ρ and the second differential r.) Taking Tot and using the isomorphism u above we get a map of complexes $\rho : \mathcal{F}([1,m]) \hat{\otimes} \mathcal{F}([m,n]) \to \mathcal{F}([1,n])$.

More generally if $I_1, \cdots, I_c$ be a segmentation of $I \subset [1,n]$, and $I_t \cap I_{t+1} = k$, there is the corresponding product map

$$\rho_k : \mathcal{F}(I_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_t) \hat{\otimes} \mathcal{F}(I_{t+1}) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c) \to \mathcal{F}(I_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_t \cup I_{t+1}) \hat{\otimes} \cdots \hat{\otimes} \mathcal{F}(I_c) \ .$$

This is defined just as above, changing the order of totalization and tensor product in the factors $\mathcal{F}(I_t)$, $\mathcal{F}(I_{t+1})$, only.

The map ρ_k is of the form $\rho_k : \mathcal{F}(I|\Sigma) \to \mathcal{F}(I|\Sigma - \{k\})$. The following diagram commutes (for distinct $k, k' \in \Sigma$):

$$\begin{array}{ccc} \mathcal{F}(I|\Sigma) & \xrightarrow{\rho_k} & \mathcal{F}(I|\Sigma - \{k\}) \\ {\scriptstyle \rho_{k'}}\downarrow & & \downarrow{\scriptstyle \rho_{k'}} \\ \mathcal{F}(I|\Sigma - \{k'\}) & \xrightarrow{\rho_k} & \mathcal{F}(I|\Sigma - \{k,k'\}) \ . \end{array}$$

For $K \subset \Sigma$ let $\rho_K : \mathcal{F}(I|\Sigma) \to \mathcal{F}(I|\Sigma - K)$ be the composition of ρ_k for $k \in K$ in any order.

(2.5.1) *Dimensions of the cycle complexes.* The dimensions of the cycle complexes can be specified as follows. To each interval $[i, i+1] \subset [1,n]$, an integer $a_i \in \mathbb{Z}$ is assigned. To a subset $I \subset [1,n]$, if $j = \mathrm{in}(I)$, $k = \mathrm{tm}(I)$, let

$$a_I = \sum_{j \le i \le k} a_i - \sum_{j \le i \le k-1} \dim X_i \ .$$

We then have the following property: If $\mathrm{tm}(I) = \mathrm{in}(I') = c$, then $a_{I \cup I'} = a_I + a_{I'} - \dim X_c$. Thus we have the map $\rho : \mathcal{F}(I) \hat{\otimes} \mathcal{F}(I') \to \mathcal{F}(I \cup I')$. We also have the map $\pi_K : \mathcal{F}(I) \to \mathcal{F}(I - K)$, to be defined in the next subsection.

(2.6) *The map π_K.* Recall for a subset $K \subset \overset{\circ}{I}$ disjoint from $\mathcal{J} \cup \Sigma$, one has the map $\pi_K : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I - K, \mathcal{J}|\Sigma)$. This is compatible with the maps $r_{\mathcal{J},\mathcal{J}'}$.

Using this we will produce, for $K \subset \overset{\circ}{I} - \Sigma$, a map of complexes $\pi_K : \mathcal{F}(I|\Sigma) \to \mathcal{F}(I - K|\Sigma)$. Let $\pi_K : \bigoplus \mathcal{F}(I, \mathcal{J}|\Sigma) \to \bigoplus \mathcal{F}(I - K, \mathcal{J}|\Sigma)$ be the sum of the maps $\pi_K : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I - K, \mathcal{J}|\Sigma)$ for K disjoint from $\mathcal{J}$, and the zero maps on $\mathcal{F}(I, \mathcal{J}|\Sigma)$ if $K \cap \mathcal{J} \neq \emptyset$. The repeated use of π_K will not lead to a confusion.

In (2.7) we summarize and and complement the properties of the complexes $\mathcal{F}(I,\mathcal{J}|\Sigma)$ and the maps ρ, π_K. In (2.8) we collect the properties of $\mathcal{F}(I|\Sigma)$ and the maps ρ, π_K. Note (2.8) rests only on the properties (2.7).

(2.7) *Properties of* $\mathcal{F}(I,\mathcal{J}|\Sigma)$ *and the maps* r, ρ, *and* π.

(0) To each $I \subset [1,n]$ and $\mathcal{J} \subset \overset{\circ}{I}$, there corresponds a complex $\mathcal{F}(I,\mathcal{J})$ of free $\mathbb{Z}$-modules. For $\mathcal{J} \subset \mathcal{J}'$, there is the corresponding quasi-isomorphism $r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}(I,\mathcal{J}) \to \mathcal{F}(I,\mathcal{J}')$. The $r_{\mathcal{J},\mathcal{J}'}$ is transitive for the inclusion $\mathcal{J} \subset \mathcal{J}' \subset \mathcal{J}''$. For $K \subset \overset{\circ}{I} - \mathcal{J}$, one has a map of complexes $\pi_K : \mathcal{F}(I,\mathcal{J}) \to \mathcal{F}(I-K,\mathcal{J})$.

There is a quasi-isomorphism $\mathbb{Z}(A_I) \to \mathcal{F}(I,\emptyset)$.

In addition, we have the following structures (1)-(4).

(1) For $\Sigma \subset \overset{\circ}{I}$ and $\mathcal{J} \subset \overset{\circ}{I} - \Sigma$, there is a quasi-isomorphic multiple subcomplex of free $\mathbb{Z}$-modules

$$\iota_\Sigma : \mathcal{F}(I,\mathcal{J}|\Sigma) \to \mathcal{F}(I,\mathcal{J}\top\Sigma) .$$

If $\Sigma = \emptyset$, then $\mathcal{F}(I,\mathcal{J}|\emptyset) = \mathcal{F}(I,\emptyset)$. The inclusion is compatible with tensor product, namely if $\Sigma \supset \Sigma'$ and Σ' gives the segmentation $I_1, \cdots, I_c$ of I, and $\mathcal{J}_i = \mathcal{J} \cap \overset{\circ}{I}_i$, then one has inclusion of c-fold complexes

$$\mathcal{F}(I,\mathcal{J}|\Sigma) \subset \mathcal{F}(I_1,\mathcal{J}_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c,\mathcal{J}_c|\Sigma_c) \tag{2.7.1}$$

where the latter group is viewed as a subgroup of $\mathcal{F}(I,\mathcal{J}\top\Sigma)$ by the tensor product of the inclusions $\mathcal{F}(I_i,\mathcal{J}_i|\Sigma_i) \subset \mathcal{F}(I_i,\mathcal{J}_i\top\Sigma_i)$.

(2) For $\mathcal{J} \subset \mathcal{J}'$, there is a quasi-isomorphism of complexes $r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}(I,\mathcal{J}|\Sigma) \to \mathcal{F}(I,\mathcal{J}'|\Sigma)$, transitive in $\mathcal{J}$. If $\Sigma = \emptyset$, it coincides with the map $r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}(I,\mathcal{J}) \to \mathcal{F}(I,\mathcal{J}')$ in (0). The map r is compatible with the inclusion (2.7.1), namely the following square commutes:

$$\begin{array}{ccc}
\mathcal{F}(I,\mathcal{J}|\Sigma) & \hookrightarrow & \mathcal{F}(I_1,\mathcal{J}_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c,\mathcal{J}_c|\Sigma_c) \\
{\scriptstyle r_{\mathcal{J},\mathcal{J}'}}\downarrow & & \downarrow{\scriptstyle \otimes r_{\mathcal{J}_i,\mathcal{J}'_i}} \\
\mathcal{F}(I,\mathcal{J}'|\Sigma) & \hookrightarrow & \mathcal{F}(I_1,\mathcal{J}'_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c,\mathcal{J}'_c|\Sigma_c) .
\end{array}$$

(3) For $K \subset \Sigma$ there is the corresponding map of complexes

$$\rho_K : \mathcal{F}(I,\mathcal{J}|\Sigma) \to \mathcal{F}(I,\mathcal{J}\cup K|\Sigma - K) .$$

If $K = K' \amalg K''$ then $\rho_K = \rho_{K''}\rho_{K'}$. If $\Sigma = \emptyset$, it coincides with the map ρ_K in (0). The map ρ_K, where K is disjoint from Σ, is compatible with the inclusion (2.7.1), namely the following diagram commutes, where $K_i = K \cap \Sigma_i$.

$$\begin{array}{ccc}
\mathcal{F}(I,\mathcal{J}|\Sigma) & \hookrightarrow & \bigoplus \mathcal{F}(I_i,\mathcal{J}_i|\Sigma_i) \\
{\scriptstyle \rho_K}\downarrow & & \downarrow{\scriptstyle \otimes \rho_{K_i}} \\
\mathcal{F}(I,\mathcal{J}\cup K|\Sigma - K) & \hookrightarrow & \bigoplus \mathcal{F}(I_i,\mathcal{J}_i\cup K_i|\Sigma_i - K_i)
\end{array}$$

(4) To $K \subset \overset{\circ}{I} - \Sigma$ disjoint from K, there corresponds the map of complexes

$$\pi_K : \mathcal{F}(I,\mathcal{J}|\Sigma) \to \mathcal{F}(I-K,\mathcal{J}|\Sigma) .$$

If $K = K' \amalg K''$ then $\pi_K = \pi_{K''}\pi_{K'}$. If $\Sigma = \emptyset$, it coincides with the map π_K in (0). The map π_K is compatible with the inclusion (2.7.1).

The maps r, ρ, and π commute with each other. The commutativity of r and ρ means the commutativity of the following square:

$$\begin{tikzcd}
\mathcal{F}(I,\mathcal{J}|\Sigma) \arrow[r, "r_{\mathcal{J},\mathcal{J}'}"] \arrow[d, "\rho_K"'] & \mathcal{F}(I,\mathcal{J}'|\Sigma) \arrow[d, "\rho_K"] \\
\mathcal{F}(I,\mathcal{J}\cup K|\Sigma-K) \arrow[r, "r_{\mathcal{J}\cup K,\mathcal{J}'\cup K}"] & \mathcal{F}(I-K,\mathcal{J}'\cup K|\Sigma-K)\,.
\end{tikzcd}$$

The reader may write down the commutative diagrams for the commutativity of r and π, and of ρ and π.

(5) The maps r, ρ and π provide another map in the derived category. Let $K\subset\Sigma$. We have the maps

$$\begin{tikzcd}
 & \mathcal{F}(I,\emptyset|\Sigma) \arrow[d, "\rho_K"] \\
\mathcal{F}(I,\emptyset|\Sigma-K) \arrow[r, "r_{\emptyset,K}"] \arrow[d, "\pi_K"'] & \mathcal{F}(I,K|\Sigma-K) \\
\mathcal{F}(I-K,\emptyset|\Sigma-K) &
\end{tikzcd}$$

Since the map r is a quasi-isomorphism, inverting it gives a map in the derived category of abelian groups

$$\varphi_K:\mathcal{F}(I,\emptyset\top\Sigma)\to\mathcal{F}(I-K,\emptyset\top\Sigma-K)\ .$$

We call this the *composition map.*

The map φ_K satisfies (a) transitivity in K, which says $\varphi_K=\varphi_{K''}\varphi_{K'}$ if $K=K'\amalg K''$, and (b) compatibility with tensor product. To state the latter, let I be partitioned by m to I' and I'', Σ be a subset containing m, and K be a subset of $\Sigma-\{m\}$. Let Σ be partitioned by m to Σ' and Σ'', and $K'=K\cap I'$, $K''=K\cap I''$. Then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{F}(I,\emptyset\top\Sigma) \arrow[r, phantom, "="] \arrow[d, "\varphi_K"'] & \mathcal{F}(I',\emptyset\top\Sigma')\otimes\mathcal{F}(I'',\emptyset\top\Sigma'') \arrow[d, "\varphi_{K'}\otimes\varphi_{K''}"] \\
\mathcal{F}(I-K,\emptyset\top\Sigma-K) \arrow[r, phantom, "="] & \mathcal{F}(I'-K',\emptyset\top\Sigma'-K')\otimes\mathcal{F}(I''-K'',\emptyset\top\Sigma''-K'')\ .
\end{tikzcd}$$

(2.8) *Properties of* $\mathcal{F}(I|\Sigma)$ *and the maps* ρ, π.

(1) $\mathcal{F}(I|\Sigma)$ is a multiple complex of free $\mathbb{Z}$-modules. For $\Sigma\subset\overset{\circ}{I}$ corresponding to the segmentation $I_1,\cdots,I_r$ of I, there is an injective quasi-isomorphism of multiple complexes

$$\iota_\Sigma:\mathcal{F}(I|\Sigma)\hookrightarrow\mathcal{F}(I\top\Sigma):=\mathcal{F}(I_1)\otimes\cdots\otimes\mathcal{F}(I_r)\ .$$

If $\Sigma\supset\Sigma'$, Σ' gives the segmentation $I_1,\cdots,I_c$ of I, and $\Sigma_i=\Sigma\cap\overset{\circ}{I_i}$, then one has inclusion

$$\mathcal{F}(I|\Sigma)\subset\mathcal{F}(I_1|\Sigma_1)\otimes\cdots\otimes\mathcal{F}(I_c|\Sigma_c) \tag{2.8.1}$$

where the latter group is viewed as a subgroup of $\mathcal{F}(I\top\Sigma)$ by the tensor product the inclusions $\iota_{\Sigma_i}:\mathcal{F}(I_i|\Sigma_i)\subset\mathcal{F}(I_i\top\Sigma_i)$.

(2) For $K\subset\Sigma$ there is a map of multiple complexes $\rho_K:\mathcal{F}(I|\Sigma)\to\mathcal{F}(I|\Sigma-K)$. If $K=K'\amalg K'$, $\rho_K=\rho_{K''}\rho_{K'}$. The ρ_K is compatible with the inclusion (2.8.1).

(3) For $K \subset \overset{\circ}{I} - \Sigma$, there is the associated map $\pi_K : \mathcal{F}(I|\Sigma) \to \mathcal{F}(I-K|\Sigma)$. If $K = K' \amalg K''$ then $\pi_K = \pi_{K''}\pi_{K'} : \mathcal{F}(I|\Sigma) \to \mathcal{F}(I-K|\Sigma)$. The map π_K is compatible with the inclusion in (2.8.1).

$\rho_{K'}$ and π_K commute with each other, namely the following square commutes:

$$\begin{array}{ccc} \mathcal{F}(I|\Sigma) & \xrightarrow{\rho_{K'}} & \mathcal{F}(I|\Sigma - K') \\ {\scriptstyle \pi_K}\downarrow & & \downarrow{\scriptstyle \pi_K} \\ \mathcal{F}(I-K|\Sigma) & \xrightarrow{\rho_{K'}} & \mathcal{F}(I-K-K'|\Sigma - K') \ . \end{array}$$

(4) $\mathcal{F}(I|\Sigma)$ is acyclic unless $\Sigma = \overset{\circ}{I}$. If $I = [1,n]$ and $I_i = [i, i+1]$,

$$\begin{aligned} \mathcal{F}(I|\overset{\circ}{I}) &= \mathcal{F}(I_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I_{n-1}) \\ &= \mathcal{F}(I_1,\emptyset)[-1]\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I_{n-1},\emptyset)[-1] \ . \end{aligned}$$

So one has quasi-isomorphisms

$$\mathcal{F}(I|\overset{\circ}{I}) \hookrightarrow \mathcal{F}(I_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I_{n-1}) \hookleftarrow \mathcal{Z}(A_{I_1})[-1]\otimes\cdots\otimes\mathcal{Z}(A_{I_{n-1}})[-1] \ .$$

(2.9) *Variant of the bar complex.* We give a variant of the bar complex. In the next subsection we will give a further variant, which will be applied to the complexes $\mathcal{F}(I|\Sigma)$.

Let $n \geq 2$ and assume:

(1) To each subset $I \subset [1,n]$ of cardinality ≥ 2, a complex of abelian groups $(A(I)^\bullet, d_A)$ is assigned.

(2) For a segmentation of I into I' and I'', there corresponds a map of complexes $\rho : A(I') \otimes A(I'') \to A(I)$. If I is segmented into three intervals I', I'', I''', then the following commutes:

$$\begin{array}{ccc} A(I')\otimes A(I'')\otimes A(I''') & \xrightarrow{\rho\otimes 1} & A(I'\cup I'')\otimes A(I''') \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ A(I')\otimes A(I''\cup I''') & \xrightarrow{1\otimes\rho} & A(I) \ . \end{array}$$

In the following we write $\alpha\cdot\beta$ for $\rho(\alpha\otimes\beta)$.

For a partition $(I_1, \cdots, I_c)$ of $[1,n]$, one has the complex $A(I_1)\otimes A(I_2)\otimes\cdots\otimes A(I_c)$. Let

$$B(A) = \bigoplus A(I_1)\otimes A(I_2)\otimes\cdots\otimes A(I_c)$$

the sum over all segmentations. Give a grading by

$$\overline{\deg}(\alpha_1\otimes\cdots\otimes\alpha_c) = \sum(\deg\alpha_i - 1)$$

and give differentials by (put $\epsilon_j = \deg(\alpha_j) - 1$)

$$\bar{d}(\alpha_1\otimes\cdots\otimes\alpha_c) = -\sum(-1)^{\sum_{j>i}\epsilon_j}\,\alpha_1\otimes\cdots\otimes\alpha_{i-1}\otimes d_A(\alpha_i)\otimes\cdots\otimes\alpha_c \ ,$$

$$\bar{\rho}(\alpha_1\otimes\cdots\otimes\alpha_c) = \sum(-1)^{\sum_{j\geq i}\epsilon_j}\,\alpha_1\otimes\cdots\otimes\alpha_{i-2}\otimes(\alpha_{i-1}\cdot\alpha_i)\otimes\cdots\otimes\alpha_c \ .$$

One verifies $\bar{d}\bar{d} = 0$, $\bar{\rho}\bar{\rho} = 0$ and $\bar{d}\bar{\rho} + \bar{\rho}\bar{d} = 0$, so that $d_{B(A)} = \bar{d} + \bar{\rho}$ is a differential. Thus with the grading $\overline{\deg}$ and the differential $d_{B(A)}$, $B(A)$ is a complex; we call it the *bar complex* associated to $(A(I); d_A, \rho)$. To specify n we may write $B([1,n]; A)$.

For a subset $\Sigma \subset (1,n)$ there corresponds a segmentation $I_1, \cdots, I_c$ of $[1,n]$. Let

$$A([1,n] \,\top\, \Sigma) = A(I_1) \otimes A(I_2) \otimes \cdots \otimes A(I_c) \ .$$

Then

$$B([1,n]; A) = \bigoplus_{\Sigma} A([1,n] \,\top\, \Sigma)$$

as a group. The differential $\bar{d}$ is the sum of $\bar{d} : A([1,n] \,\top\, \Sigma) \to A([1,n] \,\top\, \Sigma)$.

For $k \in \Sigma$ the product map induces a map

$$\bar{\rho}_k : A([1,n] \,\top\, \Sigma) \to A([1,n] \,\top\, \Sigma - \{k\}) \ ,$$

so that $\bar{\rho}$ is the sum of them.

There are quotient complexes of $B([1,n])$ defined as follows. For a subset $S \subset (1,n)$, $\bigoplus_{\Sigma \not\supset S} A([1,n] \,\top\, \Sigma)$ is a subcomplex of $B([1,n])$; the quotient complex is denoted $B([1,n] \,\top\, S)$:

$$B([1,n] \,\top\, S) = \bigoplus_{\Sigma \supset S} A([1,n] \,\top\, \Sigma) \ .$$

For $S = \emptyset$, $B([1,n] \,\top\, \emptyset) = B([1,n])$. If $S \subset S'$ there is a natural surjection of complexes

$$\tau_{S\, S'} : B([1,n] \,\top\, S) \to B([1,n] \,\top\, S') \ .$$

If $S \subset S' \subset S''$ then $\tau_{S\, S''} = \tau_{S'\, S''} \tau_{S\, S'}$.

Note the above construction can be applied to any subset $I \subset [1,n]$, $|I| \geq 2$, in place of $[1,n]$. So one has the complex $B(I)$ and, for $S \subset \overset{\circ}{I}$, the complex $B(I \,\top\, S)$.

If S corresponds to a segmentation $I_1, \cdots, I_c$ of $[1,n]$, there is a natural equality of complexes

$$B(I \,\top\, S) = B(I_1) \otimes \cdots \otimes B(I_c) \ .$$

Here the right hand side is the usual tensor product of complexes.

(2.10)*Further variant of the bar complex.* Let $n \geq 2$ and we make the following assumption. It is the same condition that the complexes $\mathcal{F}(I, \mathcal{J})$ satisfy, except there is no quasi-isomorphism with the cycle complex $\mathcal{Z}(A_I)$. Besides the complexes $\mathcal{F}(I, \mathcal{J})$, we will encounter another example in a later section.

Assumption (A)

(A-0) To each subset I of $[1,n]$ and $\mathcal{J} \subset \overset{\circ}{I}$, there corresponds a complex $A(I, \mathcal{J})$ of free $\mathbb{Z}$-modules. For $\mathcal{J} \subset \mathcal{J}'$ there is a corresponding map $r_{\mathcal{J},\mathcal{J}'} : A(I, \mathcal{J}) \to A(I, \mathcal{J}')$; the map is transitive in $\mathcal{J}$. For $K \subset \overset{\circ}{I} - \mathcal{J}$ one has a map $\pi_K : A(I, \mathcal{J}) \to A(I - K, \mathcal{J})$; one has $\pi_K = \pi_{K'} \pi_{K''}$ if $K = K' \amalg K''$.

(A-1) For $\Sigma \subset \overset{\circ}{I}$ and $\mathcal{J} \subset \overset{\circ}{I} - \Sigma$, let $A(I, \mathcal{J} \,\top\, \Sigma)$ be the tensor product $A(I_1, \mathcal{J}_1) \otimes \cdots \otimes A(I_c, \mathcal{J}_c)$, where $I_1, \cdots, I_c$ is the segmentation of I by Σ, and $\mathcal{J}_i = \mathcal{J} \cap \overset{\circ}{I_i}$. There is a quasi-isomorphic multiple subcomplex of free $\mathbb{Z}$-modules

$$\iota_\Sigma : A(I, \mathcal{J} | \Sigma) \to A(I, \mathcal{J} \,\top\, \Sigma) \ .$$

If $\Sigma = \emptyset$, then $A(I, \mathfrak{J}|\emptyset) = A(I, \emptyset)$. If $\Sigma \supset \Sigma'$ and Σ' gives the segmentation $I_1, \cdots, I_c$ of I, and $\mathfrak{J}_i = \mathfrak{J} \cap \mathring{I}_i$, then one has inclusion of c-fold complexes

$$A(I, \mathfrak{J}|\Sigma) \subset A(I_1, \mathfrak{J}_1|\Sigma_1) \otimes \cdots \otimes A(I_c, \mathfrak{J}_c|\Sigma_c) \tag{2.10.1}$$

where the latter group is viewed as a subgroup of $A(I, \mathfrak{J} \top \Sigma)$ by the tensor product of the inclusions $A(I_i, \mathfrak{J}_i|\Sigma_i) \subset A(I_i, \mathfrak{J}_i \top \Sigma_i)$.

(A-2) For $\mathfrak{J} \subset \mathfrak{J}'$, there is a quasi-isomorphism of complexes $r_{\mathfrak{J},\mathfrak{J}'} : A(I, \mathfrak{J}|\Sigma) \to A(I, \mathfrak{J}'|\Sigma)$, transitive in $\mathfrak{J}$. If $\Sigma = \emptyset$, it coincides with the map $r_{\mathfrak{J},\mathfrak{J}'} : A(I, \mathfrak{J}) \to A(I, \mathfrak{J}')$ in (0). The map r is compatible with the inclusion (2.10.1).

(A-3) For $K \subset \Sigma$ there is the corresponding map of complexes

$$\rho_K : A(I, \mathfrak{J}|\Sigma) \to A(I, \mathfrak{J} \cup K|\Sigma - K) \ .$$

If $K = K' \amalg K''$ then $\rho_K = \rho_{K''}\rho_{K'}$. If $\Sigma = \emptyset$, it coincides with the map ρ_K in (0). The map ρ_K, where K is disjoint from Σ, is compatible with the inclusion (2.10.1).

(A-4) To $K \subset \mathring{I} - \Sigma$ disjoint from K, there corresponds the map of complexes

$$\pi_K : A(I, \mathfrak{J}|\Sigma) \to A(I - K, \mathfrak{J}|\Sigma) \ .$$

If $K = K' \amalg K''$ then $\pi_K = \pi_{K''}\pi_{K'}$. If $\Sigma = \emptyset$, it coincides with the map π_K in (0). The map π_K is compatible with the inclusion (2.10.1).

The maps r, ρ, and π commute with each other.

We note that the same construction as before gives us the composition map $\varphi_K : A(I, \emptyset \top \Sigma) \to A(I - K, \emptyset \top \Sigma - K)$ in the derived category.

We constructed $\mathcal{F}(I|\Sigma)$ from $\mathcal{F}(I, \mathfrak{J}|\Sigma)$; the same procedure gives us complexes $A(I|\Sigma)$ and pertinent maps as follows.

(1) Let $I \subset [1, n]$ and $\Sigma \subset \mathring{I}$. If $\Sigma \subset \mathring{I}$ corresponds to the segmentation $I_1, \cdots, I_c$ of I, there is a quasi-isomorphic multiple subcomplex of free $\mathbb{Z}$-modules

$$\iota_\Sigma : A(I|\Sigma) \hookrightarrow A(I \top \Sigma) = A(I_1) \otimes \cdots \otimes A(I_c) \ .$$

We let the same $A(I|\Sigma)$ denote its total complex.

If $\Sigma \supset \Sigma'$, Σ' gives the segmentation $I_1, \cdots, I_c$ of I and $\Sigma_i = \Sigma \cap \mathring{I}_i$, then one has inclusion

$$A(I|\Sigma) \subset A(I_1|\Sigma_1) \otimes \cdots \otimes A(I_c|\Sigma_c)$$

where the latter group is viewed as a subgroup of $A(I \top \Sigma)$ by the the tensor product of the inclusions $\iota_{\Sigma_i} : A(I_i|\Sigma_i) \subset A(I_i \top \Sigma_i)$.

(2) For $K \subset \Sigma$ there is a map of multiple complexes $\rho_K : A(I|\Sigma) \to A(I|\Sigma - K)$. If $K = K' \amalg K''$, $\rho_K = \rho_{K''}\rho_{K'}$. The ρ_K is compatible with the inclusion $A(I|\Sigma) \subset \otimes A(I_i|\Sigma_i)$ in (1).

(3) For $K \subset \mathring{I} - \Sigma$, there is a map of multiple complexes $\pi_K : A(I|\Sigma) \to A(I - K|\Sigma)$. If $K = K' \amalg K''$ then $\pi_K = \pi_{K''}\pi_{K'} : A(I|\Sigma) \to A(I - K|\Sigma)$. π_K and $\rho_{K'}$ commute with each other, namely the following square commutes:

$$\begin{tikzcd}
A(I|\Sigma) \arrow[r, "\rho_{K'}"] \arrow[d, "\pi_K"'] & A(I|\Sigma - K') \arrow[d, "\pi_K"] \\
A(I - K|\Sigma) \arrow[r, "\rho_{K'}"] & A(I - K|\Sigma - K') \ .
\end{tikzcd}$$

(4) The complex $A(I)$ is acyclic if $|I| \geq 3$. Hence $A(I|\Sigma)$ is acyclic unless $\Sigma = \mathring{I}$.

To compare with the previous subsection, (1) and (2) are weaker assumptions than before; (3) gives additional structure, and (4) is satisfied because $A(I)$ if of the form $\bigoplus A(I, \mathfrak{J})$.

One can now define the bar complex $B([1,n]; A)$ as before. To be precise $B([1,n]; A) = \bigoplus_\Sigma A([1,n]|\Sigma)$ as a group, and the differential is given by $d_{B(A)} = \bar{d} + \bar{\rho}$, where $\bar{d}$ and $\bar{\rho}$ are defined as follows. If $I_1, \cdots, I_c$ is the partition of $[1,n]$ corresponding to Σ, for an element $\alpha = \alpha_1 \otimes \cdots \otimes \alpha_c \in A([1,n]|\Sigma)$, let $\epsilon_j = \deg(\alpha_j) - 1$ and

$$\bar{d}(\alpha_1 \otimes \cdots \otimes \alpha_c) = -\sum (-1)^{\sum_{j>i} \epsilon_j} \alpha_1 \otimes \cdots \otimes \alpha_{i-1} \otimes d_A(\alpha_i) \otimes \cdots \otimes \alpha_c \ ,$$

$$\bar{\rho}(\alpha_1 \otimes \cdots \otimes \alpha_c) = \sum_{2 \leq i \leq c} (-1)^{\sum_{j \geq i} \epsilon_j} \rho_{k_{i-1}}(\alpha)$$

with $k_{i-1} = \mathrm{tm}(I_{i-1})$.

For $S \subset (1,n)$ there is defined the corresponding quotient $B([1,n]|S)$; for $S \subset S'$ there is a natural surjection $\sigma_{S\,S'} : B([1,n]|S) \to B([1,n]|S')$. The construction applies to any subset $I \subset [1,n]$ and $S \subset \mathring{I}$, so one has $B(I)$, $B(I|S)$, and maps $\sigma_{S\,S'}$.

It follows from (4) that the maps $\sigma_{S\,S'} : B(I|S) \to B(I|S')$ are quasi-isomorphisms. Indeed $A(I|\Sigma)$ is acyclic unless $\Sigma = \mathring{I}$, so the quotient map $B(I|S) \to A(I|\mathring{I})$ is a quasi-isomorphism.

If S corresponds to the segmentation $I_1, \cdots, I_c$, let

$$B(I \top S) := B(I_1) \otimes \cdots \otimes B(I_c) \ .$$

One has an injective quasi-isomorphism

$$\iota_S : B(I|S) \hookrightarrow B(I \top S)$$

defined as the sum of the quasi-isomorphisms $A(I|\Sigma) \hookrightarrow A(I_1|\Sigma_1) \otimes \cdots \otimes A(I_c|\Sigma_c)$, where $\Sigma \supset S$ and $\Sigma_1, \cdots, \Sigma_c$ is the segmentation of Σ given by S.

For $S \subset S'$, also define the map $\tau_{S\,S'} : B(I \top S) \to B(I \top S')$ as follows. Let $\Sigma \supset S$, $\{I_i\}$ the segmentation of I by S, and $\Sigma_i = \Sigma \cap \mathring{I}_i$. In case $\Sigma \supset S'$, if $\{I'_j\}$ the segmentation of I by S' and $\Sigma'_j = \Sigma \cap \mathring{I}'_j$, there in an inclusion

$$A(I_1|\Sigma_1) \otimes \cdots \otimes A(I_c|\Sigma_c) \hookrightarrow A(I'_1|\Sigma'_1) \otimes \cdots \otimes A(I'_d|\Sigma'_d) \ .$$

Define $\tau_{S\,S'}$ to be this inclusion on the summand $A(I_1|\Sigma_1) \otimes \cdots \otimes A(I_c|\Sigma_c)$ with $\Sigma \supset S'$, and zero on the summand with $\Sigma \not\supset S'$. The maps $\tau_{S\,S'}$ and $\sigma_{S\,S'}$ are compatible, namely the following diagram commutes:

$$\begin{array}{ccc} B(I|S) & \xrightarrow{\iota_S} & B(I \top S) \\ {\scriptstyle \sigma_{S\,S'}}\downarrow & & \downarrow{\scriptstyle \tau_{S\,S'}} \\ B(I|S') & \xrightarrow{\iota_{S'}} & B(I \top S') \ . \end{array}$$

If $S \subset S' \subset S''$ then $\tau_{S\,S''} = \tau_{S'\,S''} \tau_{S\,S'}$.

For $K \subset \mathring{I}$ disjoint from S, one has a map

$$\varphi_K : B(I|S) \to B(I - K|S)$$

given as follows. Define a quotient complex of $B(I|S)$ by

$$B_K(I|S) := \bigoplus_{\substack{\Sigma \supset S \\ (\mathfrak{J} \cup \Sigma) \cap K = \emptyset}} A(I|\Sigma) \ .$$

There is a map

$$B_K(I|S) \to B(I-K|S)$$

which is the sum of $\pi_K : A(I, \mathfrak{J}|\Sigma) \to A(I-K, \mathfrak{J}|\Sigma)$. By definition φ_K is the composition of the quotient map $B(I|S) \to B_K(I|S)$ with the above map.

If $K = K' \amalg K''$ then $\varphi_K = \varphi_{K''}\varphi_{K'} : B(I|S) \to B(I-K|S)$. If K and S' are disjoint, the maps $\sigma_{S\,S'}$ and φ_K commute with each other, namely the following diagram commutes.

$$\begin{tikzcd}
B(I|S) \arrow[r, "\sigma_{S\,S'}"] \arrow[d, "\varphi_K"'] & B(I|S') \arrow[d, "\varphi_K"] \\
B(I-K|S) \arrow[r, "\sigma_{S\,S'}"] & B(I-K|S')
\end{tikzcd}$$

(2.11)*Properties of the complex $B(I|S)$.* For future reference we collect properties of the bar complex.

(1) $B(I)$ is a complex of free $\mathbb{Z}$-modules. For $S \subset \mathring{I}$ corresponding to a segmentation $I_1, \cdots, I_c$ of I, let $B(I \intercal S) = B(I_1) \otimes \cdots \otimes B(I_c)$. $B(I|S)$ is a complex of free $\mathbb{Z}$-modules together with an injective quasi-isomorphism $\iota_S : B(I|S) \hookrightarrow B(I \intercal S)$. If $S = \emptyset$, $B(I|\emptyset) = B(I)$. If $S = \mathring{I}$, $I = [1, n]$ and $I_i = [i, i+1]$,

$$\begin{aligned}
B(I|\mathring{I}) &= A(I_1)[1] \hat{\otimes} \cdots \hat{\otimes} A(I_{n-1})[1] \\
&= A(I_1, \emptyset) \hat{\otimes} \cdots \hat{\otimes} A(I_{n-1}, \emptyset) = A(I, \emptyset|\mathring{I}).
\end{aligned}$$

If $S \supset S'$, S' gives the segmentation $I_1, \cdots, I_c$ and $S_i = \mathring{I}_i \cap S$, then one has inclusion

$$B(I|S) \subset B(I_1|S_1) \otimes \cdots \otimes B(I_c|S_c) \subset B(I \intercal S) \ .$$

(2) For subsets $S \subset S'$ there corresponds a surjective quasi-isomorphism $\sigma_{S\,S'} : B(I|S) \to B(I|S')$. One has $\sigma_{S\,S''} = \sigma_{S'\,S''}\sigma_{S\,S'}$. The σ is compatible with the inclusion $B(I|S) \subset B(I_1|S_1) \otimes \cdots \otimes B(I_c|S_c)$, namely if $S \subset S''$ and $S''_i = S'' \cap \mathring{I}_i$, the following commutes:

$$\begin{tikzcd}
B(I|S) \arrow[r, hook] \arrow[d, "\sigma_{S\,S''}"'] & B(I_1|S_1) \otimes \cdots \otimes B(I_c|S_c) \arrow[d, "\otimes\sigma_{S_i\,S''_i}"] \\
B(I|S'') \arrow[r, hook] & B(I_1|S''_1) \otimes \cdots \otimes B(I_c|S''_c) \ .
\end{tikzcd}$$

There are quasi-isomorphisms (in general not surjective or injective) $\tau_{S\,S'} : B(I \intercal S) \to B(I \intercal S')$ for $S \subset S'$. One has $\tau_{S\,S''} = \tau_{S'\,S''}\tau_{S\,S'}$.

The maps $\sigma_{S\,S'}$ and $\tau_{S\,S'}$ are compatible via the maps ι_S, $\iota_{S'}$.

(3) There are maps $\varphi_K : B(I|S) \to B(I-K|S)$ which satisfy $\varphi_K = \varphi_{K''}\varphi_{K'}$ if $K = K' \amalg K''$ and are compatible with $\sigma_{S\,S'}$. The following square commutes in the derived category.

$$\begin{tikzcd}
B(I|S) \arrow[r, "\sigma"] \arrow[d, "\varphi_K"'] & A(I, \emptyset|\mathring{I}) \arrow[r, "\iota"] & A(I, \emptyset \intercal \mathring{I}) \arrow[d, "\varphi_K"] \\
B(I-K|S) \arrow[r, "\sigma"] & A(I-K, \emptyset|\mathring{I}-K) \arrow[r, "\iota"] & A(I-K, \emptyset \intercal \mathring{I}-K)
\end{tikzcd}$$

Here the right vertical map is the composition map mentioned in Assumption (A).

In addition, we have:

(2.12) **Proposition.** *Let R, J be disjoint subsets of $\overset{\circ}{I}$, with J non-empty. Then the following sequence of complexes is exact (the maps are alternating sums of the quotient maps σ)*

$$B(I|R) \xrightarrow{\sigma} \bigoplus_{S\subset J\ ,|S|=1} B(I|R\cup S) \xrightarrow{\sigma} \bigoplus_{S\subset J\ ,|S|=2} B(I|R\cup S) \xrightarrow{\sigma} \cdots \to B(I|R\cup J) \to 0\ .$$

Moreover the total complex of the sequence is acyclic. (Equivalently, the induced map σ : $B(I|R) \to \operatorname{Ker}\Big(\sigma : \bigoplus_{S\subset J\ ,|S|=1} B(I|R\cup S) \to \bigoplus_{S\subset J\ ,|S|=2} B(I|R\cup S)\Big)$ is a surjective quasi-isomorphism.)

Proof. For $\Sigma \subset \overset{\circ}{I}$ the complex $A(I|\Sigma)$ appears in $B(I|R\cup S)$ as a direct summand iff $\Sigma \supset R\cup S$. Thus the sequence in question is the direct sum over Σ of the following:

$$A(I|\Sigma) \to \bigoplus_{S\subset J\cap(\Sigma-R)\ ,|S|=1} A(I|\Sigma) \to \bigoplus_{S\subset J\cap(\Sigma-R)\ ,|S|=2} A(I|\Sigma) \to \cdots\ .$$

If $J\cap(\Sigma-R)\neq\emptyset$ this is exact, even with 0 at left. If $J\cap(\Sigma-R)=\emptyset$ this is trivially exact.

(2.13) *The complex $F(I|S)$.* With the notation in (2.1)-(2.8), we take the association of complexes $I\mapsto \mathcal{F}(I)$, together with quasi-isomorphisms $\mathcal{F}(I|\Sigma)\hookrightarrow \mathcal{F}(I \,\top\, \Sigma)$ and the maps ρ_k and π_K. Apply the construction of the bar complex. We obtain the complexes $B(I|S)$ and the maps σ, φ. We employ the notation

$$F([1,n]|S) \qquad \text{or } F(X_1,\cdots,X_n|S)$$

for this complex. We use the same letter S for the base variety and for a subset of $(1,n)$, but this should not cause confusion. Likewise for any subset I of $[1,n]$ there are the complexes $F(I|S)$.

One has $F(I|\emptyset)=F(I)$. For $S\subset S'$ there is a natural surjection $\sigma_{S\,S'} : F(I|S)\to F(I|S')$. For $K\subset \overset{\circ}{I}$ disjoint from K, there is the map $\varphi_K : F(I|S)\to F(I-K|S)$.

These complexes and the maps satisfy the properties we have proven to hold in general. In particular

$$F(I|\overset{\circ}{I}) = \mathcal{F}(I,\emptyset|\overset{\circ}{I})\ .$$

Thus the following maps are all quasi-isomorphisms (let $I=[1,n]$, $I_i=[i,i+1]$):

$$\begin{aligned} F(I) &\twoheadrightarrow F(I|\overset{\circ}{I}) = \mathcal{F}(I,\emptyset|\overset{\circ}{I}) \\ &\hookrightarrow \mathcal{F}(I,\emptyset \,\top\, \overset{\circ}{I}) \\ &\hookleftarrow \mathcal{Z}(A_{I_1})\otimes\cdots\otimes\mathcal{Z}(A_{I_{n-1}}) \end{aligned}$$

providing an isomorphism $F(I)\cong \mathcal{Z}(A_{I_1})\otimes\cdots\otimes\mathcal{Z}(A_{I_{n-1}})$ in the derived category. If $|I|=2$, $F(I)=\mathcal{F}(I,\emptyset)$.

According to (2.5.1), we must specify dimensions for the cycle complexes by assigning integers to each $[i,i+1]$, $i=1,\cdots,n-1$.

(2.14) *The quasi DG category $Symb(S)$* . Let S be a quasi-projective variety, and $Symb(S)$ be the class of symbols over S, (0.8). Recall a symbol is a finite formal sum $\bigoplus_\alpha (X_\alpha/S, r_\alpha)$, where X_α is a smooth variety with a projective map to S. There is direct sum of symbols.

There is a structure of quasi DG category on $Symb(S)$ defined as follows. For a finite sequence of symbols of the form $(X_i/S, r_i)$, let

$$F\big((X_1/S, r_1), \cdots, (X_n/S, r_n)\big)$$

be the complex $F(X_1, \cdots, X_n)$ with respect to the sequence of dimensions

$$[i, i+1] \mapsto \dim X_{i+1} - r_{i+1} + r_i \ , i = 1, \cdots, n-1.$$

For any sequence of symbols K_i, define the complex $F(K_1, \cdots, K_n)$ by linearity.

We thus have the complexes $F(K_1, \cdots, K_n|S)$, and maps $\sigma_{S\,S'}$, φ_K satisfying the properties as before.

The class of objects $Symb(S)$, together with these complexes and maps, still denoted $Symb(S)$. It will be proven in a later section that this forms a quasi DG category.

Remark. There is the structure of a category on $Symb(S)$ as in [4]. This is not used in this paper, and we refer the reader to [4]for details. Let us only say that the homomorphism group is

$$\mathrm{Hom}((X/S, r), (Y/S, s)\,) = \mathrm{CH}_{\dim Y - s + r}(X \times_S Y)$$

and the composition is to be defined appropriately.

3 Distinguished subcomplexes with respect to constraints

In §1 we discussed distinguished subcomplexes of the form $\mathcal{Z}(M_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n)$ for a sequence of fiberings M_i on $[1, n]$. There are other forms of distinguished subcomplexes of the tensor product $\mathcal{Z}(M_1) \otimes \cdots \otimes \mathcal{Z}(M_n)$. For example if M_i is a sequence of fiberings on $[1, n+1]$ and an element $f \in \mathcal{Z}(M_{n+1})$ is given, we may want to consider a subcomplex of $\mathcal{Z}(M_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{Z}(M_n)$ generated by $\alpha_1 \otimes \cdots \otimes \alpha_r$ that is properly intersecting with f. We will explain such generalizations in (3.1), (3.2).

In (3.3)-(3.5) we proceed to discuss variants where $\mathcal{Z}(M_i)$ is replaced with $\mathcal{Z}(M_i, \mathcal{U}_i)$, $\mathcal{F}(I)$ or $F(I)$. In (3.5) we consider the complex $F(I|S)$ as defined in §2, and show typically a result as follows. If J is another finite ordered set with $\mathrm{tm}(I) = \mathrm{in}(J) = c$, and $f(J|T) \in F(J|T)$ an element, there is a distinguished subcomplex $[F(I|S)]_f$ of $F(I|S)$ such that the map

$$(-) \otimes f : [F(I|S)]_f \to F(I \cup J|S \cup \{c\} \cup T)$$

is defined. In [8]we will only be concerned with $F(I|S)$ and its distinguished subcomplexes.

The rest of this section (3.6)-(3.9) has to do with a particular example of a distinguished subcomplex. Such a subcomplex appears in [8], where we construct the complex $\mathbb{F}(K_1, \cdots, K_n)$ for a sequence of diagrams K_i. So we suggest the reader to read this part only when it is needed in [8]

(3.1) We give a prototype for the distinguished subcomplexes which appear in this section.

Let $I = [1, n]$ be a sub-interval of $\mathbb{I} = [-N, N']$. Let M_i $(i = -N, \cdots, N')$ and Y_i $(-N \le i \le N' - 1)$ be smooth varieties with smooth maps $M_i \to Y_i$ and $M_{i+1} \to Y_i$, namely M_i is a sequence of fiberings indexed by $\mathbb{I}$. One has the product $M_{-N} \times \cdots \times M_{N'}$ and a subspace $M_{[-N,N']}$, the fiber product.

Assume given a set of elements $f_i \in \mathcal{Z}(M_i, m_i)$ for $i \in [-N, N'] - [1, n]$, which are irreducible, non-degenerate, satisfying the following condition: The set $\{f_i \ (i \in [-N, N'] - [1, n]), \text{faces}\}$ is properly intersecting in $M_{[-N,N']} \times \square^*$. Define the quasi-isomorphic subcomplex

$$[\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n)]_{\{f_i\}}$$

to be the subcomplex of $\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n)$ generated by elements $\alpha_1 \otimes \cdots \otimes \alpha_n$, with each α_i irreducible, such that the set

$$\{\alpha_1, \cdots, \alpha_n, \quad f_i \ (i \in [-N, N'] - [1, n]), \text{faces}\}$$

is properly intersecting in $M_{[-N,N']}$.

We show this is a distinguished subcomplex. Indeed consider the set of cycles

$$V = \{M_{[-N,N']}, f_i \ (i \in [-N, N'] - [1, n])\}$$

in $M_{-N} \times \cdots \times M_{N'}$; together with the faces it is properly intersecting. By the lemma in (1.7), the required condition is equivalent to $\{\alpha_1, \cdots, \alpha_n, V, \text{faces}\}$ being properly intersecting in $M_{-N} \times \cdots \times M_{N'}$. Thus the complex is of the type of Example in (1.6.1).

The following properties are obvious from the definitions.

(1) For each m with $n \le m \le N'$, one has a similarly defined complex $[\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_m)]_{\{f_i\}}$; to be precise one uses only the cycles f_i with $i \in [-N, N'] - [1, m]$. There is a map

$$(-) \otimes f_{m+1} : [\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_m)]_f \to [\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_{m+1})]_f$$

that sends $\alpha_1 \otimes \cdots \otimes \alpha_m$ to $\alpha_1 \otimes \cdots \otimes \alpha_m \otimes f_{m+1}$. The same holds for the map $f_i \otimes (-)$, $i < 0$.

(2) If $I_1, \cdots, I_r$ is a partition of $[1, n]$, the product induces a map

$$\rho(I_1, \cdots, I_r) : [\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n)]_f \to [\mathcal{Z}(M_{I_1}) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_{I_r})]_f \ .$$

(3.2) To generalize the above it will be convenient to state the relevant structure of the cycle complex as axioms. Axioms (a)-(c) are evidently satisfied for the cycle complex. Axiom (d) consists of the existence of distinguished subcomplexes, that are generalizations of the above prototype.

(3.2.1) *Distinguished subcomplex with respect to a constraint.* The complex $\mathcal{Z}(M, \bullet)$ has the following structure.

(a)(set of generators) There is a set $\mathcal{S}(M, m)$ such that $\mathcal{Z}(M, m)$ is free on $\mathcal{S}(M, m)$. (Specifically it is the set of irreducible non-degenerate admissible cycles.) $\mathcal{S}(M, m)$ is additive in M, namely if $M = M' \amalg M''$, then $\mathcal{S}(M, m) = \mathcal{S}(M', m) \amalg \mathcal{S}(M'', m)$.

(b)(notion of proper intersection) Let M_i be a sequence of fiberings indexed by $[1, n]$. If A is a subset of $[1, n]$ and $\{\alpha_i \in \mathcal{S}(M_i, m_i) \mid i \in A\}$ is a set of elements indexed by A, we are given whether or not the set $\{\alpha_i | \ i \in A\}$ is *properly intersecting*. (Instead of saying $\{\alpha_i\,, \text{faces}\}$ is properly intersecting, we may just say $\{\alpha_i\}$ is properly intersecting.) We have:

- If $\{\alpha_i |\ i \in A\}$ is properly intersecting, for any subset B of A, $\{\alpha_i |\ i \in B\}$ is properly intersecting.
- Let A and A' be subsets such that $\mathrm{tm}(A) + 1 < \mathrm{in}(A')$ (A and A' are not adjacent). If $\{\alpha_i |\ i \in A\}$ and $\{\alpha_i |\ i \in A'\}$ are properly intersecting sets indexed by A and A' respectively, the union $\{\alpha_i |\ i \in A \cup A'\}$ is also properly intersecting.
- If $\{\alpha_1, \cdots, \alpha_n\}$ is properly intersecting, then for any i, writing $\partial\alpha_i = \sum c_{i\nu}\beta_\nu$ with $\beta_\nu \in S(M_i, m_i - 1)$, each set

$$\{\alpha_1, \cdots, \alpha_{i-1}, \beta_\nu, \alpha_{i+1}, \cdots, \alpha_n\}$$

is properly intersecting. In other words, the notion of proper intersection is compatible with ∂.
- Assume $M_i = M_i' \amalg M_i''$ and $\alpha_i \in \mathcal{S}(M_i')$ for $i \in A$. Then $\{\alpha_i \in \mathcal{S}(M_i) | i \in A\}$ is properly intersecting if and only if $\{\alpha_i \in \mathcal{S}(M_i') | i \in A\}$ is properly intersecting.

Let $\mathcal{Z}(M_1)\hat{\otimes} \cdots \hat{\otimes}\mathcal{Z}(M_n)$ be the submodule generated by $\alpha_1 \otimes \cdots \otimes \alpha_n$, where $\alpha_i \in \mathcal{S}(M_i, m_i)$ and $\{\alpha_1, \cdots, \alpha_n\}$ is properly intersecting. This is a subcomplex by the third property. It is additive in each variable M_i.

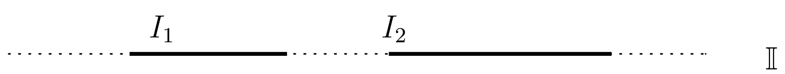
(c)(product map) When $\{\alpha_1, \cdots, \alpha_n\}$ with $\alpha_i \in \mathcal{S}(M_i, m_i)$ is properly intersecting, the product $\alpha_1 \circ \cdots \circ \alpha_n \in \mathcal{Z}(M_1 \diamond \cdots \diamond M_n, m_1 + \cdots m_n)$ is defined. For this product, we have:

- The product gives a map of complexes $\rho : \hat{\otimes}\mathcal{Z}(M_i) \to \mathcal{Z}(M_1 \diamond \cdots \diamond M_n)$.
- More generally if $I_1, \cdots, I_r$ is a partition of $[1, n]$, and $\alpha_{I_j} \in \mathcal{Z}(M_{I_j})$ is the product of α_i's for $i \in I_j$, then the set $\{\alpha_{I_1}, \cdots, \alpha_{I_r}\}$ is properly intersecting. Further the resulting map $\rho(I_1, \cdots, I_r) : \hat{\otimes}\mathcal{Z}(M_i) \to \hat{\otimes}\mathcal{Z}(M_{I_j})$ is a map of complexes.
- The product $\rho(I_1, \cdots, I_r)$ satisfies associativity as in (1.8).

(d)(distinguished subcomplexes) Let I be a finite (totally) ordered set, and $(M_i)_{i\in I}$ be a collection of smooth varieties indexed by I (we do not assume given a sequence of varieties on I). We will consider distinguished subcomplexes of $\bigotimes_{i\in I} \mathcal{Z}(M_i)$ obtained specifically as follows. The basic type is (d-1). By taking tensor products and finite intersections we get (d-2) and (d-3).

(d-1) Let $\mathbb{I}$ be a finite ordered set and $I \hookrightarrow \mathbb{I}$ an inclusion. The image of I need not be a sub-interval of $\mathbb{I}$. Then there is a partition $I_1, \cdots, I_r$ of I such that

- The image of each I_a is a sub-interval of $\mathbb{I}$.
- For each a, $\mathrm{tm}(I_a) + 1 < \mathrm{in}(I_{a+1})$ (I_a are not adjacent to each other).



Assume given a sequence of fiberings M_i indexed by $\mathbb{I}$, extending the given M_i on I. Specifically we must give M_i for $i \in \mathbb{I}$, Y_i for $i \in \mathbb{I} - \{\mathrm{tm}(\mathbb{I})\}$ and maps from M to Y.

Let $f = (f_j)$ be a set of properly intersecting elements $f_j \in \mathcal{S}(M_j, m_j)$, where j varies over a subset A of $\mathbb{I} - I$. Let I' be a subset of I. The set of data consisting of

$$I \hookrightarrow \mathbb{I}; \quad M \text{ on } \mathbb{I}; \quad I'; \quad f = (f_j)$$

is called a *constraint* (the set $f_j \in \mathcal{S}(M_j, m_j)$ itself is also called a constraint). Then the subcomplex generated by $\otimes_{i \in I} \alpha_i$, where the set

$$\{\alpha_i \, (i \in I'), \quad f_j \, (j \in A)\,\}$$

is properly intersecting, is a quasi-isomorphic subcomplex of $\bigotimes_{i \in I} \mathcal{Z}(M_i)$. This subcomplex is denoted

$$[\bigotimes_{i \in I} \mathcal{Z}(M_i)]_{\mathbb{I}, I'; f} \ ,$$

or $[\bigotimes_{i \in I} \mathcal{Z}(M_i)]_f$, and called the *distinguished subcomplex* with respect to $(\mathbb{I}, I'; f)$, or $\{f\}$.

If $I = [1, n]$, the image of I is a sub-interval, $I' = I$ and f is empty (namely A is empty) then the corresponding subcomplex is just $\hat{\otimes}_{i \in I} \mathcal{Z}(M_i)$. For the prototype discussed before, $I = [1, n]$, $\mathbb{I} = [-N, N']$, $A = \mathbb{I} - I$, and $I' = I$. Generalizing the notation for the prototype case, if $I' = I$, the subcomplex is written

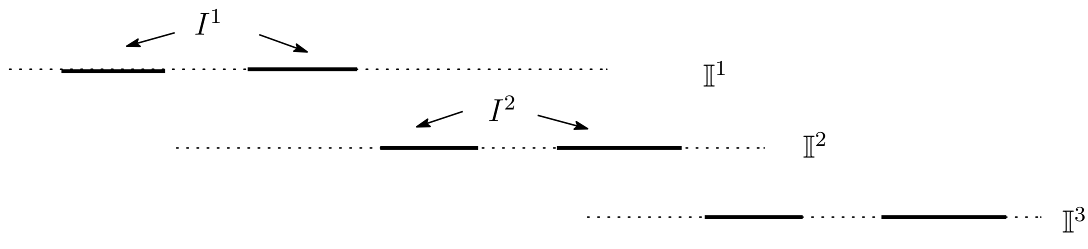
$$[\widehat{\bigotimes_{i \in I_1}} \mathcal{Z}(M_i) \otimes \widehat{\bigotimes_{i \in I_2}} \mathcal{Z}(M_i) \otimes \cdots \otimes \widehat{\bigotimes_{i \in I_r}} \mathcal{Z}(M_i)]_f \ .$$

(The hat over I_a indicates the cycles α_i for $i \in I_a$ are properly intersecting.) If in addition f is empty, it coincides with $\widehat{\bigotimes}_{i \in I_1} \mathcal{Z}(M_i) \otimes \widehat{\bigotimes}_{i \in I_2} \mathcal{Z}(M_i) \otimes \cdots \otimes \widehat{\bigotimes}_{i \in I_r} \mathcal{Z}(M_i)$.

(d-2) One can consider tensor products of subcomplexes in (d-1), as follows. Let $I^1, \cdots, I^s$ be a partition of I. For each k assume given a finite ordered set $\mathbb{I}^k$ and an inclusion $I^k \hookrightarrow \mathbb{I}^k$, a sequence of varieties M_i^k indexed by $\mathbb{I}^k$, extending the given M_i on I^k, properly intersecting elements $f^k = \{f_j^k \in \mathcal{S}(M_j^k, m_j^k) \mid j \in A^k \subset \mathbb{I}^k - I^k\}$, and a subset $(I^k)' \subset I^k$. The set of data

$$\{(I^1, \cdots, I^s); \quad I^k \hookrightarrow \mathbb{I}^k; \quad M^k \text{ on } \mathbb{I}^k; \quad (I^k)'; \quad f^k\}_k$$

is called a constraint. Note that there is no imposed relation between $\mathbb{I}^k$'s for distinct k's. The image of I^k in $\mathbb{I}^k$ need not be a sub-interval. If $s = 1$ the data is the same as in (d-1).



Then the subcomplex of $\bigotimes_{i \in I} \mathcal{Z}(M_i)$ generated by $\otimes_{i \in I} \alpha_i$, where for each k the set

$$\{\alpha_i \ (i \in (I^k)'\,), \quad f_j^k \ (j \in A^k)\,\}$$

is properly intersecting, is a quasi-isomorphic subcomplex. If the collection $(\mathbb{I}^k)$ is denoted by $\mathbb{I}$, (f^k) by f, $((I^k)'\,)$ by I', then the subcomplex may be denoted $[\bigotimes_{i \in I} \mathcal{Z}(M_i)]_{\mathbb{I}, I'; f}$. Since there is no interaction between I^k's, the subcomplex coincides with the tensor product

$$[\bigotimes_{i \in I^1} \mathcal{Z}(M_i)]_{\mathbb{I}^1, (I^1)'; f^1} \otimes \cdots \otimes [\bigotimes_{i \in I^s} \mathcal{Z}(M_i)]_{\mathbb{I}^s, (I^s)'; f^s} \ .$$

(d-3) The intersection of a finite number of subcomplexes of type (d-2) is a distinguished subcomplex.

For subcomplexes of type (d-1) it is described as follows. For each $\nu = 1, \cdots, c$, let $I \hookrightarrow \mathbb{I}(\nu)$ be an inclusion into a finite ordered set. Let $M(\nu)_i$ be an extension of M_i to $\mathbb{I}(\nu)$, $f(\nu) = (f(\nu)_j)$ be properly intersecting elements where $j \in A(\nu) \subset \mathbb{I}(\nu) - I$, and $I(\nu)' \subset I$ a subset. No relation is imposed between the data for distinct ν. One thus has a finite set of constraints

$$\{I \hookrightarrow \mathbb{I}(\nu); \quad M(\nu) \text{ on } \mathbb{I}(\nu); \quad I(\nu)'; \quad f(\nu)\}_\nu \ .$$

For each ν one has the distinguished subcomplex $[\bigotimes_{i\in I} \mathcal{Z}(M_i)]_{\mathbb{I}(\nu),I(\nu)';f(\nu)}$; the intersection

$$\bigcap_\nu [\bigotimes_{i\in I} \mathcal{Z}(M_i)]_{\mathbb{I}(\nu),I(\nu)';f(\nu)}$$

is again a quasi-isomorphic subcomplex, and called the distinguished subcomplex with respect to the finite set of constraints.

We can do the same for subcomplexes of type (d-2). For each $\nu = 1, \cdots, c$, consider a constraint: a partition $I(\nu)^1, \cdots, I(\nu)^{s(\nu)}$ of I, and for each $k = 1, \cdots, s(\nu)$,

$$I(\nu)^k \hookrightarrow \mathbb{I}(\nu)^k; \quad \text{an extension } M(\nu)^k \text{ of } M \text{ to } \mathbb{I}^k; \quad (I(\nu)^k)' \subset I(\nu)^k; \quad f(\nu)^k = (f(\nu)^k_j) \ .$$

Take the corresponding distinguished subcomplex, and then take the intersection for ν. The resulting subcomplex is still a distinguished subcomplex. This is the most general type of distinguished subcomplexes in (d). It is still denoted by $[\bigotimes_{i\in} \mathcal{Z}(M_i)]_{\mathbb{I},I';f}$.

One shows the tensor product of complexes of type (d-3) is again of the same type. So it is the smallest class of subcomplexes containing (d-1), and closed under taking tensor product and finite intersections.

By a distinguished subcomplex (with respect to a constraint) we mean any one of type (d), especially (d-3).

(e)(properties) It is evident from the definition that subcomplexes in (d) have the following properties.

- In case (d-1), for $j \in A$ one has a map

$$(-) \otimes f_j : [\bigotimes_{i\in I} \mathcal{Z}(M_i)]_{\mathbb{I},I';f} \to [\bigotimes_{i\in I\cup\{j\}} \mathcal{Z}(M_i)]_{\mathbb{I},I'\cup\{j\};f}$$

  that sends $\otimes_{i\in I} \alpha_i$ to $(\otimes_{i\in I} \alpha_i) \otimes f_j$. Similarly for the cases (d-2) and (d-3).

- In case (d-1), if $I' = I$, I is a sub-interval of $\mathbb{I}$ (namely $r = 1$), and $J_1, \cdots, J_s$ is a partition of I, the product induces a map

$$\rho(J_1, \cdots, J_s) : [\widehat{\bigotimes_{i\in I}} \mathcal{Z}(M_i)]_f \to [\widehat{\bigotimes_{i}} \mathcal{Z}(M_{J_i})]_f \ .$$

  More generally assume I' is a sub-interval of $\mathbb{I}$ and $J_1, \cdots, J_s$ a partition of I'. Let $\bar{I} = (I - I') \cup \{1, \cdots, s\}$ be the finite ordered set obtained from I by replacing I' by $\{1, \cdots, s\}$; it parametrizes the set of varieties M_i for $i \in I - I'$ and M_{J_j} for $j = 1, \cdots, s$.

If $\bar{\mathbb{I}} = (\mathbb{I} - I') \cup \{1, \cdots, s\}$ is the finite ordered set obtained from $\mathbb{I}$ in a similar manner, there is an injection $\bar{I} \hookrightarrow \bar{\mathbb{I}}$, and there is a sequence of varieties on $\bar{\mathbb{I}}$ extending $\{M_i, M_{J_j}\}$. There is the product map (product within I')

$$\rho(J_1, \cdots, J_s) : [\bigotimes_{i \in I} \mathcal{Z}(M_i)]_{\mathbb{I}, I'; f} \to [\bigotimes_{i \in I - I'} \mathcal{Z}(M_i) \otimes \bigotimes_j \mathcal{Z}(M_{J_j})]_{\bar{\mathbb{I}}, \{1, \cdots, s\}; f}$$

Similarly for the cases (d-2) and (d-3).

(3.2.2) *Generalizations of properly intersecting sets.* In (3.2.1)(b) we discussed the condition of proper intersection for $\alpha_i \in \mathcal{S}(M_i, m_i)$. Here are some generalizations.

(1) For a set of elements $\alpha_i \in \mathcal{Z}(M_i, m_i)$, $i = 1, \cdots, n$, let us say the set $\{\alpha_1, \cdots, \alpha_n\}$ is properly intersecting if the following condition is satisfied: Let A be set of i such that $\alpha_i \neq 0$. For $i \in A$ write $\alpha_i = \sum c_{i\nu} \alpha_{i\,\nu}$ with $\alpha_{i\,\nu}$ irreducible non-degenerate. Then for any choice of ν_i for $i \in A$, the set

$$\{\alpha_{i\,\nu_i} \mid i \in A\}$$

is properly intersecting.

(2) Let $L_1, \cdots, L_b$ be disjoint intervals of $[1, n]$, and $\alpha_j \in \widehat{\bigotimes}_{i \in L_j} \mathcal{Z}(M_i, m_i)$ for $j = 1, \cdots, b$. Writing each α_j as a sum of tensors of elements in $\mathcal{S}(M_i, m_i)$, one can define the condition of proper intersection for the set $\{\alpha_1, \cdots, \alpha_b\}$.

(3) Let $J_1, \cdots, J_s$ be disjoint intervals of $[1, n]$, and $\alpha_i \in \mathcal{Z}(M_{J_i}, m_i)$. One can define for $\{\alpha_1, \cdots, \alpha_s\}$ the condition of proper intersection. More generally, assume each J_i is partitioned into $J_{i1}, \cdots, J_{i\,k_i}$; then for a set of elements $\alpha_i \in \widehat{\bigotimes}_j \mathcal{Z}(M_{J_{i\,j}})$, $i = 1, \cdots, s$, one can define the condition of proper intersection.

(3.2.3) *Generalizations of constraints.* Now that the notion of proper intersection has been generalized, we can also generalize the notion of constraints and the corresponding distinguished subcomplexes. For simplicity consider only the type (d-1), but one can do the same for (d-2) and (d-3).

(1) Keep the notation of (d-1). Let $J_j \subset \mathbb{I} - I$, $j = 1, \cdots, s$ be a disjoint set of intervals and $f_j \in \mathcal{Z}(M_{J_j})$ be a properly intersecting set of elements. One can then form the corresponding distinguished subcomplex.

(2) More generally, let $J_j \subset \mathbb{I} - I$, $j = 1, \cdots, s$ be a disjoint set of intervals in $\mathbb{I} - I$, where each J_j is partitioned into $J_{j1}, \cdots, J_{j\,k_i}$. Let $f_j \in \widehat{\bigotimes}_\lambda \mathcal{Z}(M_{J_{j,\lambda}})$, $j = 1, \cdots, s$, be a properly intersecting set of elements. One has the corresponding distinguished subcomplex.

In all these variants the distinguished subcomplexes are denoted $[\bigotimes_{i \in I} \mathcal{Z}(M_i)]_{\mathbb{I}, I'; f}$.

(3.3) *Distinguished subcomplexes of $\mathcal{Z}(M_1, \mathcal{U}_1) \otimes \cdots \otimes \mathcal{Z}(M_n, \mathcal{U}_n)$ with respect to constraints.* Let M and $\mathcal{U}$ be as in (1.2). We can repeat all of (3.2) for the complex $\mathcal{Z}(M, \mathcal{U})$. Since $\mathcal{Z}(M, \mathcal{U}) = \bigoplus_I \mathcal{Z}(U_I)$, where I varies over subsets of the indexing set of $\mathcal{U}$, an element $\alpha \in \mathcal{Z}(M, \mathcal{U})$ is of the form $\sum_I \alpha_I$ with $\alpha_I \in \mathcal{Z}(U_I)$. There is a filtration of $\mathcal{Z}(M, \mathcal{U})$ by subcomplexes such that the successive quotients are direct sums of $\mathcal{Z}(U_I)$.

Since $\mathcal{Z}(U_I)$ is $\mathbb{Z}$-free on the set $\mathcal{S}(U_I)$, $\mathcal{Z}(M, \mathcal{U})$ is free on

$$\mathcal{S}(M, \mathcal{U}) := \amalg_I \mathcal{S}(U_I) \ .$$

Let M_i be a sequence of fiberings indexed by $[1,n]$, and let $\mathcal{U}_i$ be a finite covering of $U_i \subset M_i$. For elements $\alpha_i \in \mathcal{S}(M_i, \mathcal{U}_i)$, i varying over a subset $A \subset [1,n]$, we have defined in §1 when $\{\alpha_i\}$ is properly intersecting. The properties in (3.2.1)(b) are satisfied.

When α_i, $i = 1, \cdots, n$ are properly intersecting the product $\alpha_1 \circ \cdots \circ \alpha_n \in \mathcal{Z}(M_1 \diamond M_2 \diamond \cdots \diamond M_n, \mathcal{U}_1 \amalg \cdots \amalg \mathcal{U}_n)$ is defined. The properties in (3.2.1)(c) are satisfied.

One can proceed as in (3.2.1)(d), except one replaces $\mathcal{Z}(M_i)$ with $\mathcal{Z}(M_i, \mathcal{U}_i)$, to define distinguished subcomplexes of tensor product $\bigotimes \mathcal{Z}(M_i, \mathcal{U}_i)$ with respect to a constraint. For example, as in (d-1), one can define a distinguished subcomplex of the form

$$[\bigotimes_{i \in I} \mathcal{Z}(M_i, \mathcal{U}_i)]_{\mathbb{I}, I'; f}$$

where $f_j \in \mathcal{S}(M_j, \mathcal{U}_j)$, $j \in A \subset \mathbb{I} - I$, is a properly intersecting set. One shows this is a quasi-isomorphic subcomplex of $\bigotimes_{i \in I} \mathcal{Z}(M_i, \mathcal{U}_i)$ by considering a filtration and reducing to the case (3.2.1).

Generalization of proper intersection (3.2.2) and of constraints (3.2.3) can be given in the same manner.

(3.4) *Distinguished subcomplexes of $\mathcal{F}(I|\Sigma)$ with respect to constraints.* From §2 recall $\mathcal{F}(I) = \oplus \mathcal{F}(I, \mathcal{J})$. Since $\mathcal{F}(I, \mathcal{J}) = \mathcal{Z}(X_I^{\mathcal{J}}, \mathcal{U}(\mathcal{J}))$ is $\mathbb{Z}$-free on $\mathcal{S}(X_I^{\mathcal{J}}, \mathcal{U}(\mathcal{J}))$, $\mathcal{F}(I)$ is $\mathbb{Z}$-free on

$$\mathcal{S}_{\mathcal{F}}(I) := \coprod_{\mathcal{J}} \mathcal{S}(X_I^{\mathcal{J}}, \mathcal{U}(\mathcal{J}))\ .$$

There is a filtration on $\mathcal{F}(I)$ by subcomplexes such that the successive quotients are direct sums of $\mathcal{F}(I, \mathcal{J})$ as complexes. To show the subcomplexes appearing in (3.4.1) and (3.4.2) below are quasi-isomorphic subcomplexes, we use this filtration and reduce to the case $\mathcal{Z}(X_I^{\mathcal{J}}, \mathcal{U}(\mathcal{J}))$.

To a segmentation of $I = [1,n]$ into sub-intervals $I_1, \cdots, I_r$, and a set of subsets $\mathcal{J}_i \subset \mathring{I}_i$, there corresponds a sequence of fiberings consisting of $X_{I_1}^{\mathcal{J}_1}, \cdots, X_{I_r}^{\mathcal{J}_r}$. (In this subsection all intervals are of cardinality ≥ 2.) For simplicity we often write X_I for $X_I^{\mathcal{J}}$. If $i_k = \mathrm{tm}\, I_k$, the sequence looks like:

$$\begin{tikzcd}
X_{I_1} \arrow[dr] & & X_{I_2} \arrow[dl] \arrow[dr] & & \cdots & & X_{I_r} \arrow[dl] \\
& X_{i_1} & & {} & & X_{i_{r-1}} &
\end{tikzcd}\ .$$

More generally, let $I_1, \cdots, I_r$ be sub-intervals of I such that $\mathrm{tm}(I_i) \leq \mathrm{in}(I_{i+1})$ for each i (then we say that the set $\{I_i\}$ is *almost disjoint*). We can complement it to a segmentation of I by adding sub-intervals of cardinality 2, $[j, j+1]$, not contained in any I_i. There corresponds a sequence of fiberings consisting of X_{I_i} and $X_{[j,j+1]}$. So for elements $\alpha_i \in \mathcal{S}_{\mathcal{F}}(I_i)$, one has the condition for the set $\{\alpha_1, \cdots, \alpha_r\}$ be properly intersecting on $X_I^{\mathcal{J}}$, where $\mathcal{J} = \cup \mathcal{J}_i$. The properties (3.2.1)(b) are satisfied.

If $I_1, \cdots, I_r$ is a segmentation of I, and $\{\alpha_i \in \mathcal{S}_{\mathcal{F}}(I_i)\}$ is a properly intersecting set, the product $\alpha_1 \circ \cdots \circ \alpha_r \in \mathcal{F}(I)$ is defined. The properties (3.2.1)(c) are satisfied with obvious changes in notation.

What we will describe in the rest of this subsection is a repetition of (3.2.1)(d) in this setting. There is to be no essential change, but notation appears different. We start with the counterpart of the subcomplex $\mathcal{Z}(M_1) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(M_n)$.

(3.4.1) **Definition.** For a segmentation $I_1, \cdots, I_c$ of I, let

$$\mathcal{F}(I_1)\hat{\otimes}\mathcal{F}(I_2)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I_r)$$

be the quasi-isomorphic subcomplex of $\mathcal{F}(I_1) \otimes \mathcal{F}(I_2) \otimes \cdots \otimes \mathcal{F}(I_r)$ generated by elements $\alpha_1 \otimes \cdots \otimes \alpha_r$, where $\alpha_i \in \mathcal{S}_{\mathcal{F}}(I_i)$ are properly intersecting.

If $\Sigma \subset (1, n)$ is the subset corresponding to the segmentation, we also write $\mathcal{F}([1, n]|\Sigma)$ for the distinguished subcomplex. The same definitions apply to any subset I of $[1, n]$.

This definition coincides with the one in (2.5), which is $\mathcal{F}(I|\Sigma) = \bigoplus_{\mathcal{J}} \mathcal{F}(I, \mathcal{J}|\Sigma)$.

According to (3.2.2), the notion of proper intersection can be generalized as follows. Let $I_1, \cdots, I_r$ be almost disjoint in I, and $\Sigma_i \subset \overset{\circ}{I_i}$. For elements $\alpha_i \in \mathcal{F}(I_i|\Sigma_i)$, $i = 1, \cdots, r$, one has the condition of proper intersection.

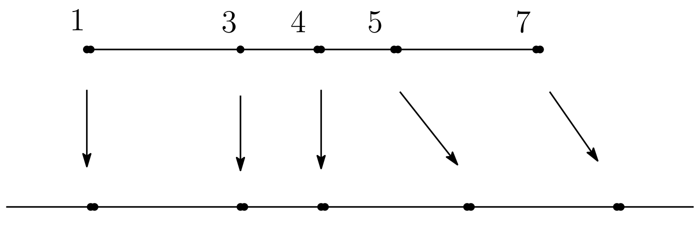
(3.4.2) Let I be a finite ordered set, $L_1, \cdots, L_r$ be almost disjoint sub-intervals such that $\cup L_i = I$; equivalently, $\mathrm{in}(L_1) = \mathrm{in}(I)$, $\mathrm{tm}(L_i) = \mathrm{in}(L_{i+1})$ or $\mathrm{tm}(L_i)+1 = \mathrm{in}(L_{i+1})$, and $\mathrm{tm}(L_r) = \mathrm{tm}(I)$. Assume given a sequence of varieties X_i on I. Consider the complex $\mathcal{F}(L_1)\otimes\cdots\otimes\mathcal{F}(L_r)$. Following (3.2.1)(d), we give the definition of its distinguished subcomplexes.

(d-1) This corresponds to (3.2.1)(d-1). First note there are subcomplexes described as follows. Let $I_1, \cdots, I_c$ be a set of almost disjoint sub-intervals of I with union I, that is coarser than $L_1, \cdots, L_r$; this means each I_a is a union of L_i's, and if I_a, $I_{a+1} \subset L_i$, then $\mathrm{tm}(I_a) = \mathrm{in}(I_{a+1})$. Then there are subsets $\Sigma_i \subset \overset{\circ}{I_i}$ such that the segmentations of I_i by Σ_i, when combined for all i, give precisely the L_i's. For our convenience we call such $I_1, \cdots, I_c$ a *regrouping* of $L_1, \cdots, L_r$. Then the complex $\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)$ is a distinguished subcomplex of $\mathcal{F}(L_1)\otimes\cdots\otimes\mathcal{F}(L_r)$. The coarser the regrouping is, the smaller the corresponding subcomplex is. If I_a and I_{a+1} satisfy $\mathrm{tm}(I_a) = \mathrm{in}(I_{a+1}) = t$, then replacing I_a, I_{a+1} by $I_a \cup I_{a+1}$ gives another regrouping, then the corresponding subcomplex

$$\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_a \cup I_{a+1}|\Sigma_a \cup \{t\} \cup \Sigma_{a+1}) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)$$

is a subcomplex of $\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)$.

Let $I \hookrightarrow \mathbb{I}$ be an inclusion into another finite ordered set $\mathbb{I}$ such that the image of each I_a is a sub-interval; we say the inclusion is compatible with $(I_1, \cdots, I_c)$. For example, let $I = [1, 7]$, $I_1 = [1, 3]$, $I_2 = [3, 4]$, $I_3 = [5, 7]$. Let $\mathbb{I} = [0, 9]$ and $I \hookrightarrow \mathbb{I}$ be defined by $i \mapsto i$ for $i \leq 4$, and $i \mapsto i + 1$ for $i \geq 5$.



Assume given an extension of X to $\mathbb{I}$. Let $J_1, \cdots, J_s \subset \mathbb{I}$ be sub-intervals of $\mathbb{I}$ such that the set $\{I_i, J_j\}_{i,j}$ is almost disjoint, and $f_j \in \mathcal{F}(J_j)$, $j = 1, \cdots, s$ be a properly intersecting set. Then one can define the distinguished subcomplex

$$[\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)]_{\mathbb{I};f} \ .$$

It is the subcomplex generated by $\alpha_1 \otimes \cdots \otimes \alpha_c$, $\alpha_i \in \mathcal{F}(I_i|\Sigma_i)$, such that $\{\alpha_1, \cdots, \alpha_c, f_j\,(j = 1, \cdots, s)\,\}$ is properly intersecting.

It is obvious to see this is a special case of (3.2.1)(d-1). From X on I we obtain a sequence of fiberings consisting of X_{L_i} and $X_{[j,j+1]}$ for $[j, j+1] \subset I - \cup L_i$; this extends to a sequence consisting of X_{L_i} and $X_{[j,j+1]}$ for $[j, j+1] \subset \mathbb{I} - \cup L_i$. The regrouping specifies the set I' in (3.2.1)(d-1).

Note that according to (3.2.2) the constraint can be generalized as follows. If $T_j \subset \overset{\circ}{J_j}$ are subsets, one may take properly intersecting elements $f_j \in \mathcal{F}(J_j|T_j)$.

(d-2) Tensor products of subcomplexes of type (d-1) are again of the same form. First we note tensor products of complexes of the form $\mathcal{F}(L_1) \otimes \cdots \otimes \mathcal{F}(L_r)$ are again of the same form. Let I' be another finite ordered set, $L'_1, \cdots, L'_{r'}$ almost disjoint sub-intervals with union I'. Let $I \cup I'$ denote the disjoint union of I and I', where $i < i'$ if $i \in I$, $i' \in I'$, and let X be a sequence of varieties on $I \cup I'$. Then $L_1, \cdots, L_r, L'_1, \cdots, L'_{r'}$ are almost disjoint sub-intervals with union $I \cup I'$. The corresponding complex is the tensor product

$$\mathcal{F}(L_1) \otimes \cdots \otimes \mathcal{F}(L_r) \otimes \mathcal{F}(L'_1) \otimes \cdots \otimes \mathcal{F}(L'_{r'}) \ .$$

To describe tensor products of complexes of type (d-1), let $I^1, \cdots, I^s$ be almost disjoint sub-intervals of I with union I.

For each k assume given the following data. Let $I^k_1, \cdots, I^k_{c_k}$ be almost disjoint sub-intervals of I^k such that $\cup I^k_i = I^k$. Each I^k_i is assumed to be a union of some of L_a's. Let $\mathbb{I}^k$ be another finite ordered set, and $I^k \hookrightarrow \mathbb{I}^k$ be an embedding compatible with $(I^k_1, \cdots, I^k_{c_k})$. On $\mathbb{I}^k$ given a sequence of varieties X^k_i that extends X on I^k. For distinct k, there is no relation between X^k's.

Given also sub-intervals $J^k_j \subset \mathbb{I}^k$ such that $\{I^k_i, J^k_j\}$ is almost disjoint in $\mathbb{I}^k$, and properly intersecting elements $f^k_j \in \mathcal{F}(J^k_j|T^k_j)$, where $T^k_j \subset (J^k_j)^\circ$. Let $\Sigma^k_i \subset (I^k_i)^\circ$ be subsets such that the segmentations of I^k_i by Σ^k_i, when combined for all k, i, give precisely L_a's.

Then the distinguished subcomplex of the following form is defined:

$$\big[\bigotimes_{k=1,\cdots,s} \big(\mathcal{F}(I^k_1|\Sigma^k_1) \otimes \cdots \otimes \mathcal{F}(I^k_{c_k}|\Sigma^k_{c_k})\big)\big]_{\mathbb{I};f} \ .$$

This is no other than a tensor product of distinguished subcomplexes of type (d-1).

(d-3) One can take finite intersections of subcomplexes of type (d-2):

With the notation in (d-2), we fix I and L_a's, and X_i. We let vary the choices of the following data: sub-intervals I^k; and for each k sub-intervals I^k_i, inclusion $I^k \hookrightarrow \mathbb{I}^k$, extension X^k to $\mathbb{I}^k$, sub-intervals J^k_j and elements f^k_j.

The subcomplex satisfies the following properties (we restrict to the case (d-1) for simplicity).

Properties. (1) The $[\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)]_f$ is a quasi-isomorphic subcomplex of $\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)$. If $J = J_\nu$ satisfies $\mathrm{tm}(I_i) = \mathrm{in}(J) = c$ and $\mathrm{tm}(J) < \mathrm{in}(I_{i+1})$, then one has a map

$$(-)\otimes f(J|T) : [\mathcal{F}(I_1|\Sigma_1)\otimes\cdots\otimes\mathcal{F}(I_c|\Sigma_c)]_f \to [\mathcal{F}(I_1|\Sigma_1)\otimes\cdots\otimes\mathcal{F}(I_i \cup J|\Sigma_i \cup \{c\} \cup T)\otimes\cdots\otimes\mathcal{F}(I_c|\Sigma_c)]_f \ ;$$

similarly if $\mathrm{tm}(I_i) < \mathrm{in}(J)$ and $\mathrm{tm}(J) = \mathrm{in}(I_{i+1})$. If $\mathrm{tm}(I_i) = \mathrm{in}(J) = c$ and $\mathrm{tm}(J) = \mathrm{in}(I_{i+1}) = c'$, one has

$$\begin{aligned}(-)\otimes \quad f(J|T) \quad &: [\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)]_f \\ \to \quad &[\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_i \cup J \cup I_{i+1}|\Sigma_i \cup \{c\} \cup T \cup \{c'\} \cup \Sigma_{i+1}) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)]_f \ .\end{aligned}$$

(2) If $\Sigma_k \supset \Sigma'_k$, there is the corresponding product map

$$\rho : [\mathcal{F}(I_1|\Sigma_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma_c)]_f \mapsto [\mathcal{F}(I_1|\Sigma'_1) \otimes \cdots \otimes \mathcal{F}(I_c|\Sigma'_c)]_f \ .$$

(3.5) *Distinguished subcomplexes of $F(I|S)$ with respect to a constraint.* Keep the same notation from the previous subsection. According to the definition in §2, $F(I) = \bigoplus_\Sigma \mathcal{F}(I|\Sigma)$, where Σ varies over subsets of $\overset{\circ}{I}$.

Recall $\mathcal{F}(I)$ is $\mathbb{Z}$-free on $\mathcal{S}_\mathcal{F}(I)$. So $\mathcal{F}(I|\Sigma)$ is $\mathbb{Z}$-free on $\mathcal{S}_\mathcal{F}(I|\Sigma)$, the subset of $\mathcal{S}_\mathcal{F}(I_1) \times \cdots \times \mathcal{S}_\mathcal{F}(I_r)$ consisting of $(\alpha_1, \cdots, \alpha_r)$ which are properly intersecting. Thus $F(I)$ is free on the set

$$\mathcal{S}_F(I) := \coprod_\Sigma \mathcal{S}_\mathcal{F}(I|\Sigma) \ .$$

We can repeat (3.4) with $\mathcal{F}(I)$ replaced with $F(I)$.

If $I_1, \cdots, I_r$ is an almost disjoint set of sub-intervals of $I = [1, n]$, and $\alpha_i \in \mathcal{S}_F(I_i)$, one has the condition of proper intersection for $\{\alpha_1, \cdots, \alpha_r\}$. The properties (3.2.1)(b) are satisfied with obvious changes. Unlike for $\mathcal{F}(I)$ there is no product $\alpha_1 \circ \cdots \circ \alpha_r$.

(3.5.1) **Definition.** For a segmentation $I_1, \cdots, I_c$ of I, let

$$F(I_1)\hat{\otimes}F(I_2)\hat{\otimes}\cdots\hat{\otimes}F(I_r)$$

be the quasi-isomorphic subcomplex of $F(I_1) \otimes F(I_2) \otimes \cdots \otimes F(I_r)$ generated by elements $\alpha_1 \otimes \cdots \otimes \alpha_r$, where $\alpha_i \in \mathcal{S}_F(I_i)$ is a set of properly intersecting elements. If $S \subset (1, n)$ is the subset corresponding to the segmentation, we also write $F(I|S)$ for the distinguished subcomplex.

The complex is equal to

$$\bigoplus \mathcal{F}(I_1|\Sigma_1)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I_r|\Sigma_r)$$

the sum over $\Sigma_i \subset \overset{\circ}{I_i}$. Since each summand equals $\mathcal{F}(I|\Sigma)$, where $\Sigma = (\cup\Sigma_i) \cup S$, one has $F(I|S) = \bigoplus_{\Sigma \supset S} \mathcal{F}(I|\Sigma)$, which agrees with the definition of $F(I|S)$ given in §2.

(3.5.2) One can repeat (3.4.2). Let I be a finite ordered set, $L_1, \cdots, L_r$ be almost disjoint sub-intervals such that $\cup L_i = I$; equivalently, $\mathrm{in}(L_1) = \mathrm{in}(I)$, $\mathrm{tm}(L_i) = \mathrm{in}(L_{i+1})$ or $\mathrm{tm}(L_i)+1 = \mathrm{in}(L_{i+1})$, and $\mathrm{tm}(L_r) = \mathrm{tm}(I)$. Assume given a sequence of varieties X_i on I. Consider the complex $F(L_1) \otimes \cdots \otimes F(L_r)$. Below we only discuss its subcomplexes of type (d-1).

Let $I_1, \cdots, I_c$ be a set of almost disjoint sub-intervals of I with union I, that is coarser than $L_1, \cdots, L_r$; let $S_i \subset \overset{\circ}{I_i}$ such that the segmentations of I_i by S_i, when combined for all i, give precisely the L_i's. Let $I \hookrightarrow \mathbb{I}$ be an inclusion into a finite ordered set $\mathbb{I}$ such that the image of each I_a is a sub-interval. Assume given an extension of X to $\mathbb{I}$. Let $J_1, \cdots, J_s \subset \mathbb{I}$ be sub-intervals of $\mathbb{I}$ such that the set $\{I_i, J_j\}_{i,j}$ is almost disjoint, and $f_j \in F(J_j|T_j)$, $j = 1, \cdots, s$ be a properly intersecting set. Then one can define the distinguished subcomplex

$$[F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c)]_{\mathbb{I};f} \ .$$

It is the subcomplex generated by $\alpha_1 \otimes \cdots \otimes \alpha_c$, $\alpha_i \in F(I_i|S_i)$, such that $\{\alpha_1, \cdots, \alpha_c, f_j \, (j = 1, \cdots, s) \}$ is properly intersecting.

The discussions for tensor products and finite intersections are parallel to (3.4.2). We have the same properties as Property (1) in (3.4).

(3.6) *Variant of (3.2).* We explain a particular example of (3.2) in steps (A) to (C). The rest of this section will be used only in Par II.

(A) This is a special case of (3.2.1)(d-1), but we describe it for clarity. Let $\mathbb{I}$ be a finite ordered set, and I a sub-interval. (Recall in (3.2.1)(d-1), I need not be a sub-interval.) Assume given a sequence of fiberings M_i indexed by $\mathbb{I}$. Note one has the space $M_{\mathbb{I}}$, the fiber product of M_i for $i \in \mathbb{I}$.

Assume given, for each interval $J \subset \mathbb{I} - I$ of cardinality ≥ 1, an element $f(J) \in \mathcal{Z}(M_J, m_J)$; they are subject to the following condition: For any disjoint set of intervals $J_1, \cdots, J_a$ contained in $\mathbb{I} - I$, the set

$$\{f(J_1), \cdots, f(J_a), \text{faces}\}$$

is properly intersecting in $M_{\mathbb{I}} \times \square^*$.
Then the subcomplex of $\widehat{\bigotimes}_{i \in I} \mathcal{Z}(M_i)$ generated by $\otimes_{i \in I} \alpha_i$ satisfying the following condition is distinguished: For each disjoint set of intervals $J_1, \cdots, J_a$ contained in $\mathbb{I} - I$, the set

$$\{\alpha_i \, (i \in I), \quad f(J_1), \cdots f(J_a), \text{faces}\}$$

is properly intersecting in $M_{\mathbb{I}}$. The subcomplex is denoted $[\widehat{\bigotimes}_{i \in I} \mathcal{Z}(M_i)]_{\mathbb{I};f}$ or $[\widehat{\bigotimes}_{i \in I} \mathcal{Z}(M_i)]_f$.

If J satisfies $\mathrm{tm}(I) + 1 = \mathrm{in}(J)$, there is a map

$$(-) \otimes f(J) : [\widehat{\bigotimes_{i \in I}} \mathcal{Z}(M_i)]_f \to [\widehat{\bigotimes_{i \in I}} \mathcal{Z}(M_i) \hat{\otimes} \mathcal{Z}(M_J)]_f$$

that sends $\otimes_{i \in I} \alpha_i$ to $\otimes_{i \in I} \alpha_i \otimes f(J)$. The target is the distinguished subcomplex of the same kind associated to the sequence consisting of M_i for $i \in I$ and M_J. More precisely let $I \cup \{J\}$ be the finite ordered set obtained by adjoining to I a single point J; any element of I is smaller than J. Let $\mathbb{I}/J$ be the finite ordered set obtained from $\mathbb{I}$ by contracting J to a point. Then $I \cup \{J\}$ is a sub-interval of $\mathbb{I}/J$. There is a sequence of varieties indexed by $\mathbb{I}/J$, in which J corresponds to M_J. To $J' \subset \mathbb{I}/J - (I \cup \{J\})$ there corresponds $f(J') \in \mathcal{Z}(M_{J'})$. Then the target complex is of the form $[\widehat{\bigotimes}_I \mathcal{Z}(M_i) \hat{\otimes} \mathcal{Z}(M_J)]_{\mathbb{I}/J;f}$.

If J satisfies $\mathrm{tm}(J) + 1 = \mathrm{in}(I)$, one has a similar map $f(J) \otimes (-)$.

As in (3.2), one can generalize the notion of constraint and take elements $f(J) \in \widehat{\bigotimes}_\lambda \mathcal{Z}(M_{J_\lambda})$, where J_λ is a partition of J.

If $I_1, \cdots, I_r$ is a partitioned of I, there is the product map

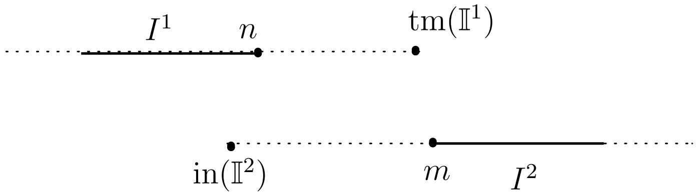
$$\rho : [\widehat{\bigotimes_{i \in I}} \mathcal{Z}(M_i)]_f \to [\widehat{\bigotimes_{i=1,\cdots,r}} \mathcal{Z}(M_{I_i})]_f \ .$$

(B) Let $I^1 = [1, n]$ and $I^2 = [m, \ell]$ be sub-intervals of $\mathbb{I}^1$, $\mathbb{I}^2$, respectively. Assume given are:

- a sequence of fiberings M_i on $\mathbb{I}^1$: $(M_i \to Y_i \leftarrow M_{i+1})$, and
- a sequence of fiberings L_i on $\mathbb{I}^2$: $(L_i \to Z_i \leftarrow L_{i+1})$.

If $n < m$ assume that $(n, m) \subset \mathbb{I}^1 \cap \mathbb{I}^2$, namely $m - 1 \leq \mathrm{tm}(\mathbb{I}^1)$, $\mathrm{in}(\mathbb{I}^2) \leq n + 1$, and (M_i) and (L_i) coincide on (n, m) as a sequence of fiberings, namely $M_i = L_i$ for $i \in (n, m)$, $Y_i = Z_i$ for

$i \in [n, m-1]$, and the projections coincide. In the following figure a solid line segment (resp. dotted line segment) represents I (resp. $\mathbb{I}$).



One can then define another sequence of fiberings $(\tilde{M}_i, \tilde{Y}_i)$ on $\tilde{\mathbb{I}} := [\mathrm{in}(\mathbb{I}^1), \mathrm{tm}(\mathbb{I}^2)]$ by

$$\tilde{M}_i = \begin{cases} M_i & \text{if } i \le n, \\ M_i = L_i & \text{if } n < i < m, \\ L_i & \text{if } i \ge m. \end{cases}$$

We call it the *glueing* of M_i and L_i along (n, m). The subset $I^1 \cup I^2$ of $\tilde{\mathbb{I}}$ is an interval if $m = n + 1$. Note in this case the condition $M_i = L_i$ on (n, m) is vacuous.

In addition, assume given a constraining set of cycles, which consists of:

• for each interval $J \subset \mathbb{I}^1 - I^1$ an element $f(J) \in \mathcal{Z}(M_J)$, and for $J \subset \mathbb{I}^2 - I^2$ an element $g(J) \in \mathcal{Z}(L_J)$. As in Step(A) one may take $f(J) \in \widehat{\bigotimes}_\lambda \mathcal{Z}(M_{J_\lambda})$ where $\{J_\lambda\}$ is a partition of J. For simplicity, though, we assume in the following $f(J) \in \mathcal{Z}(M_J)$. The general case is left to the reader.

We require:

(i) For a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\mathbb{I}^1 - I^1$, the set

$$\{f(J_\nu)\ (\nu = 1, \cdots, a), \text{ faces}\}$$

is properly intersecting in $M_{\mathbb{I}^1}$. Similar condition with respect to L_i and $g(J)$.

(ii) If $n < m$, there is a further condition. We say an interval J is between I^1 and I^2 if $n < \mathrm{in}(J) \le \mathrm{tm}(J) < m$; for such J we require $f(J) = g(J) \in \mathcal{Z}(M_J)$. Let J be an interval contained in $\tilde{\mathbb{I}} - (I^1 \cup I^2)$; then J is either to the left of I^1, between I^1 and I^2, or to the right of I^2. Define $\tilde{f}(J) \in \mathcal{Z}(\tilde{M}_J)$ by

$$\tilde{f}(J) = \begin{cases} f(J) & \text{if } J \text{ is to the left of } I^1, \\ f(J) = g(J) & \text{if } J \text{ is between } I^1 \text{ and } I^2, \\ g(J) & \text{if } J \text{ is to the right of } I^2. \end{cases}$$

The set $\{\tilde{f}(J)\}$ is the *glueing* of $\{f(J)\}$ and $\{g(J)\}$.

We also require: For a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\tilde{\mathbb{I}} - (I^1 \cup I^2)$, the set $\{\tilde{f}(J_\nu)\ (\nu = 1, \cdots, a), \text{ faces}\}$ is properly intersecting in $\tilde{M}_{\tilde{\mathbb{I}}}$.

Given such data we will define a quasi-isomorphic complex

$$[\widehat{\bigotimes_{I^1}} \mathcal{Z}(M_i) \tilde{\otimes} \widehat{\bigotimes_{I^2}} \mathcal{Z}(L_j)]_{\mathbb{I}^1, \mathbb{I}^2; f, g}$$

of the complex $\widehat{\bigotimes}_{i \in I^1} \mathcal{Z}(M_i) \otimes \widehat{\bigotimes}_{j \in I^2} \mathcal{Z}(L_j)$. It is generated by $(\alpha_1 \otimes \cdots \otimes \alpha_n) \otimes (\beta_m \otimes \cdots \otimes \beta_\ell)$, with each α_i or β_j irreducible non-degenerate, such that the following conditions are satisfied:

(i) For a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\mathbb{I}^1 - I^1$, the set

$$\{\alpha_1, \cdots, \alpha_n, \quad f(J_\nu)\ (\nu = 1, \cdots, a), \text{ faces}\}$$

is properly intersecting in $M_{\mathbb{I}^1}$. Similar condition for β's and $g(J)$.

(ii) Assume $n < m$. For a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\tilde{\mathbb{I}} - (I^1 \cup I^2)$, the set $\{\alpha_1, \cdots, \alpha_n, \beta_m, \cdots, \beta_\ell, \quad \tilde{f}(J_\nu)\ (\nu = 1, \cdots, a), \text{faces}\}$ is properly intersecting in $\tilde{M}_{\tilde{\mathbb{I}}}$. In case $n \geq m$, there is no condition (ii).

This is a distinguished subcomplex of the form (3.2.1)(d-3). The subcomplex, denoted $\mathcal{Z}(I^1; I^2)_{f,g}$ for short, has the following properties.

(1) $\mathcal{Z}(I^1; I^2)_{f,g}$ is contained in $\mathcal{Z}(I^1; I^2) = \widehat{\bigotimes}_{I^1} \mathcal{Z}(M_i) \otimes \widehat{\bigotimes}_{I^2} \mathcal{Z}(L_i)$. There are three cases according to $n = m - 1$, $n \geq m$, or $n < m - 1$. If $m = n + 1$, $\mathcal{Z}(I^1; I^2)_{f,g}$ is contained in

$$[\hat{\otimes}_{I^1 \cup I^2} \mathcal{Z}(\tilde{M}_i)]_{\tilde{f}} \ ,$$

the distinguished subcomplex of $\hat{\otimes}_{I^1 \cup I^2} \mathcal{Z}(\tilde{M}_i) = \mathcal{Z}(M_1) \hat{\otimes} \cdots \otimes \mathcal{Z}(M_n) \hat{\otimes} \mathcal{Z}(L_{n+1}) \hat{\otimes} \cdots \hat{\otimes} \mathcal{Z}(L_\ell)$ with respect to the constraint $\{\tilde{f}(J)\}$. If $n \geq m$,

$$\mathcal{Z}(I^1; I^2)_{f,g} = [\widehat{\bigotimes_{I^1}} \mathcal{Z}(M_i)]_f \otimes [\widehat{\bigotimes_{I^2}} \mathcal{Z}(L_i)]_g \ .$$

(2) Assume that J satisfies $\mathrm{tm}(I^1) + 1 = \mathrm{in}(J)$, but not necessarily that it lies between I^1 and I^2. Then one has the map

$$(-) \otimes f(J) \otimes id : \mathcal{Z}(I^1; I^2)_{f,g} \to \mathcal{Z}(I^1 \cup \{J\}; I^2)_{f,g}$$

which sends $(\otimes_{i \in I^1} \alpha_i) \otimes (\otimes_{j \in I^2} \beta_j)$ to $(\otimes_{i \in I^1} \alpha_i \otimes f(J)) \otimes (\otimes_{j \in I^2} \beta_j)$. To explain the target, $I^1 \cup \{J\}$ is the finite ordered set which is the disjoint union of I^1 and one point $\{J\}$ (J is viewed as a point); it is regarded as a sub-interval of $\mathbb{I}^1/J$, the finite ordered set obtained from $\mathbb{I}^1$ by contracting J to a single point. There is a sequence of varieties on $\mathbb{I}^1/J$, in which J corresponds to M_J. To $(I' \cup \{J\} \hookrightarrow \mathbb{I}/J, M; f)$ and $(I^2 \hookrightarrow \mathbb{I}^2, L, g)$ we may associate the distinguished subcomplex $\mathcal{Z}(I^1 \cup \{J\}; I^2)_{\mathbb{I}^1/J, \mathbb{I}^2; f, g}$. Note the target group $\mathcal{Z}(I^1 \cup J; I^2)_{f,g}$ may be of type $n \geq m$ in the classification in (1).

If J satisfies $\mathrm{tm}(J) = \mathrm{in}(I^1) - 1$, one has

$$f(J) \otimes (-) \otimes id : \mathcal{Z}(I^1; I^2)_{f,g} \to \mathcal{Z}(\{J\} \cup I^1; I^2)_{f,g} \ .$$

Similarly one has the maps $id \otimes g(J) \otimes (-) : \mathcal{Z}(I^1; I^2)_{f,g} \to \mathcal{Z}(I^1; \{J\} \cup I^2)_{f,g}$ and $id \otimes (-) \otimes g(J) : \mathcal{Z}(I^1; I^2)_{f,g} \to \mathcal{Z}(I^1; I^2 \cup \{J\})_{f,g}$.

(3) Given a partition of I^1 (or I^2) there is the product map, as in (A).

(C) For a further generalization assume given, for $k = 1, \cdots, c$,

- finite ordered set $\mathbb{I}^k$ and a sub-interval I^k, and
- a sequence of fiberings M_i^k on $\mathbb{I}^k$,

satisfying the following condition: If $n = \mathrm{tm}(I^k) < m = \mathrm{in}(I^{k+1})$, one has $(n, m) \subset \mathbb{I}^k \cap \mathbb{I}^{k+1}$, and the sequences M^k, M^{k+1} coincide on (n, m).

Note if $r < s$ and $\mathrm{tm}(I^k) < \mathrm{in}(I^{k+1})$ for $k = r, \cdots, s-1$, one can "glue" M^k, $k = r, \cdots, s$ to another sequence of fiberings $\tilde{M}$ on $\tilde{\mathbb{I}} := [\mathrm{in}(\mathbb{I}^r), \mathrm{tm}(\mathbb{I}^s)]$. One has inclusion $I^r \cup \cdots \cup I^s \subset \tilde{\mathbb{I}}$.

As a constraint, given a collection of cycles

• for each interval $J \subset \mathbb{I}^k - I^k$, an element $f^k(J) \in \mathcal{Z}(M^k_J)$,

satisfying the following condition:

(i) For a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\mathbb{I}^k - I^k$, the set

$$\{f^k(J_\nu)\ (\nu = 1, \cdots, a), \text{ faces}\}$$

is properly intersecting in $M^k_{\mathbb{I}^k}$.

(ii) If $n = \mathrm{tm}(I^k) < m = \mathrm{in}(I^{k+1})$ and J is an interval between I^k and I^{k+1}, then $f^k(J) = f^{k+1}(J) \in \mathcal{Z}(M^k_J)$.

If $r < s$ and $\mathrm{tm}(I^k) < \mathrm{in}(I^{k+1})$ for $k = r, \cdots, s-1$, one can glue $f^k(J)$ so that for each $J \subset \tilde{\mathbb{I}} - (I^r \cup \cdots \cup I^s)$ there corresponds $\tilde{f}(J) \in \mathcal{Z}(\tilde{M}_J)$. We require, for a disjoint set of intervals $J_1, \cdots, J_a$ contained in $\tilde{\mathbb{I}} - (I^r \cup \cdots \cup I^s)$, the set $\{\tilde{f}(J_\nu), \text{faces }\}$ is properly intersecting in $\tilde{M}_{\tilde{\mathbb{I}}}$.

We will define a distinguished subcomplex denoted

$$[\widehat{\bigotimes_{I^1}} \mathcal{Z}(M^1_i) \tilde{\otimes} \widehat{\bigotimes_{I^2}} \mathcal{Z}(M^2_i) \tilde{\otimes} \cdots \tilde{\otimes} \widehat{\bigotimes_{I^c}} \mathcal{Z}(M^c_i)]_{f^1, \cdots, f^c}\ ,$$

or $\mathcal{Z}(I^1; \cdots; I^c)_f$ for short. It is generated by tensors

$$(\otimes_{i \in I^1} \alpha^1_i) \otimes \cdots \otimes (\otimes_{i \in I^c} \alpha^c_i), \quad \alpha^k_i \in \mathcal{Z}(M^k_i)\,,$$

satisfying the following condition. Let $\underline{\alpha}^k = \{\alpha^k_i \mid i \in I^k\}$.

• For each pair $r \le s$ as in (ii) above, and a set of intervals $J_1, \cdots, J_a$ contained in $\tilde{\mathbb{I}} - (I^r \cup \cdots \cup I^s)$, the set

$$\{\underline{\alpha}^r, \cdots, \underline{\alpha}^s, \tilde{f}(J_\nu), \text{ faces}\}$$

is properly intersecting in $\tilde{M}_{\tilde{\mathbb{I}}}$. (If $r = s = k$, the condition reads: For each k, the set $\{\underline{\alpha}^k,\ \ f(J_\nu)\ , \text{faces}\}$ is properly intersecting in $M^k_{\mathbb{I}^k}$.) The subcomplex is distinguished, and satisfies the following properties.

Properties. (1) $\mathcal{Z}(I^1; \cdots; I^c)_f$ is a subcomplex of $\widehat{\bigotimes}_{I^1} \mathcal{Z}(M^1_i) \otimes \widehat{\bigotimes}_{I^2} \mathcal{Z}(M^2_i) \otimes \cdots \otimes \widehat{\bigotimes}_{I^c} \mathcal{Z}(M^c_i)$. For $r < s$ as in (ii) above, if $I^r \cup \cdots \cup I^s$ is an interval,

$$\mathcal{Z}(I^1; \cdots; I^c)_f \subset \mathcal{Z}(I^1; \cdots; I^{r-1}; I^r \cup \cdots \cup I^s; \cdots; I^c)_{\tilde{f}}\ .$$

The latter is the distinguished subcomplex associated with the intervals $I^k \subset \mathbb{I}^k$ for $k \ne r, \cdots, s$, $I^r \cup \cdots \cup I^s \subset \tilde{\mathbb{I}}$, the sequences M^k for $k \ne r, \cdots, s$ and $\tilde{M}$, and the constraint consisting of $f^k(J)$, $k \ne r, \cdots, s$, and $\tilde{f}(J)$.

(2) For an interval J with $\mathrm{tm}(I^k) + 1 = \mathrm{in}(J)$, one has the map

$$\mathcal{Z}(I^1; \cdots; I^c)_f \to \mathcal{Z}(I^1; \cdots; I^k \cup \{J\}; \cdots; I^c)_f$$

which sends $(\otimes_{i \in I^1} \alpha^1_i) \otimes \cdots \otimes (\otimes_{i \in I^c} \alpha^c_i)$ to $(\otimes_{i \in I^1} \alpha^1_i) \otimes \cdots \otimes (\otimes_{i \in I^k} \alpha^k_i \otimes f(J)) \otimes \cdots \otimes (\otimes_{i \in I^c} \alpha^c_i)$. Similarly for the operation $f(J) \otimes (-)$ on the k-th spot.

(3) Given a partition of I^k, there is the corresponding product map.

(3.7) *Variant of (3.3).* We have variants of (3.3), as (3.6) for (3.2). We have only to replace $\mathcal{Z}(M_i)$ with $\mathcal{Z}(M_i, \mathcal{U}_i)$. In Step (C), one has a distinguished subcomplex of the form

$$[\widehat{\bigotimes_{I^1}} \mathcal{Z}(M^1_i, \mathcal{U}^1_i) \tilde{\otimes} \widehat{\bigotimes_{I^2}} \mathcal{Z}(M^2_i, \mathcal{U}^2_i) \tilde{\otimes} \cdots \tilde{\otimes} \widehat{\bigotimes_{I^c}} \mathcal{Z}(M^c_i, \mathcal{U}^c_i)]_{f^1, \cdots, f^c}\ ,$$

having the same properties as in (3.6).

(3.8) *Variant of (3.4).* In the setting of (3.4), the variant of (3.7) can be described as follows. Assume given:

• For each $k = 1, \cdots, c$, finite ordered set $\mathbb{I}^k$, a sub-interval $I^k \subset \mathbb{I}^k$, a subset $\Sigma^k \subset \overset{\circ}{I}{}^k$, and a sequence of smooth varieties $X^k_\bullet$ indexed by $\mathbb{I}^k$.
It is required that if $n = \mathrm{tm}(I^k) \le m = \mathrm{in}(I^{k+1})$, then $[n, m] \subset \mathbb{I}^k \cap \mathbb{I}^{k+1}$ and X^k_i and X^{k+1}_i coincide on $[n, m]$. If $r < s$ and $\mathrm{tm}(I^k) \le \mathrm{in}(I^{k+1})$ for $k = r, \cdots, s-1$, one can glue $X^k_\bullet$ for $k = r, \cdots, s$ to obtain another sequence of smooth varieties $\tilde{X}$ indexed by $\tilde{\mathbb{I}} := \mathbb{I}^r \cup \cdots \cup \mathbb{I}^s$. One has $I^r \cup \cdots \cup I^s \subset \tilde{\mathbb{I}}$. The set $I^r \cup \cdots \cup I^s$ is an interval if $\mathrm{tm}(I^k) = \mathrm{in}(I^{k+1})$ for $k = r, \cdots, s-1$.

• For each interval $J \subset \mathbb{I}^k - \overset{\circ}{I}{}^k$ and a subset $T \subset \overset{\circ}{I}$ there is given an element $f^k(J|T) \in \mathcal{F}(J|T) = \mathcal{F}(J|T; X^k_\bullet)$.
If $n = \mathrm{tm}(I_k) \le m = \mathrm{in}(I_{k+1})$ and J is between I^k and I^{k+1}, then require $f^k(J|T) = f^{k+1}(J|T) \in \mathcal{F}(J|T; X^k_\bullet)$. If $r < s$ and $\mathrm{tm}(I^k) < \mathrm{in}(I^{k+1})$ for $k = r, \cdots, s-1$, one can glue $f^k(J|T)$ so that for each $J \subset \tilde{\mathbb{I}} - (\overset{\circ}{I}{}^r \cup \cdots \cup \overset{\circ}{I}{}^s)$ there corresponds an element $\tilde{f}(J|T) \in \mathcal{F}(J|T)$. We require that for any almost disjoint set of intervals $J_1, \cdots, J_a$ in $\tilde{\mathbb{I}} - (\overset{\circ}{I}{}^r \cup \cdots \cup \overset{\circ}{I}{}^s)$ and $T_\nu \subset \overset{\circ}{J}_\nu$, the set $\{\tilde{f}(J_\nu|T_\nu) \quad (\nu = 1, \cdots, a), \text{ faces }\}$ is properly intersecting in $X_{\mathbb{I}}$.

One can then define a distinguished subcomplex of the form

$$[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_{\mathbb{I};f} \ .$$

Properties. (1) $[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_f$ is a quasi-isomorphic subcomplex of $\mathcal{F}(I^1|\Sigma^1) \otimes \cdots \otimes \mathcal{F}(I^c|\Sigma^c)$. For $r < s$, if in addition $I^r \cup \cdots \cup I^s$ is an interval, then

$$[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_f$$

coincides with

$$[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^r \cup \cdots \cup I^s|\Sigma^r \cup \cdots \cup \Sigma^s \cup \{\mathrm{tm}(I^r), \cdots, \mathrm{tm}(I^{s-1})\}\,)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_{\tilde{f}} \ .$$

(2) If $J \subset I^k - \overset{\circ}{\mathbb{I}}{}^k$ with $\mathrm{tm}(I^k) = \mathrm{in}(J) = c$, then $(-) \otimes f^k(J|T)$ on the k-th factor gives a map

$$[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_f \to [\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^k \cup J|\Sigma^k \cup \{c\} \cup T)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_f \ .$$

Similarly for $f^k(J|T) \otimes (-)$ on the k-th factor.

(3) If $\Sigma^k \supset \Sigma'^k$, there is the corresponding product map

$$[\mathcal{F}(I^1|\Sigma^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma^c)]_f \to [\mathcal{F}(I^1|\Sigma'^1)\tilde{\otimes} \cdots \tilde{\otimes}\mathcal{F}(I^c|\Sigma'^c)]_f \ .$$

(3.9) *Variant of (3.5).* One can do the same as (3.8), with $\mathcal{F}(I|\Sigma)$ replaced with $F(I|S)$. One thus has a distinguishes subcomplex of the form

$$[F(I^1|S^1)\tilde{\otimes} \cdots \tilde{\otimes}F(I^c|S^c)]_f$$

where $S^k \subset \overset{\circ}{I}{}^k$, and f^k is a set of elements $f^k(J|T) \in F(J|T; X^k_\bullet)$. One has the same properties as in Properties (1), (2) in (3.8).

4 The diagonal cycle and the diagonal extension

We keep the notation of §2. We abbreviate $X_I^{\mathcal{J}}$ to X_I, and $U_I^{\mathcal{J}}$ to U_I. The map $r_{\mathcal{J},\mathcal{J}'} : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I, \mathcal{J}'|\Sigma)$ is written r_k if $\mathcal{J}' = \mathcal{J} \cup \{k\}$.

(4.1) *The diagonal cycles* $\mathbf{\Delta}(I)$. Let X be a smooth variety, projective over S, and $X_i = X$ be a constant sequence of varieties on $[1, n]$. There is the diagonal embedding $\Delta : X \to X \times_S \cdots \times_S X$; denote the image of the fundamental class of X by $\Delta(1, \cdots, n) \in \mathcal{Z}(X \times_S \cdots \times_S X)$. There is a natural quasi-isomorphism

$$\iota : \mathcal{Z}(X \times_S \cdots \times_S X) \to \mathcal{Z}(X_{[1,n]}, \{U_{[1,n]}\}) = \mathcal{F}([1, n], \emptyset) \ .$$

We use the same $\Delta(1, \cdots, n)$ to denote its image under this map. It thus consists of $\Delta(1, \cdots, n)$ in $\mathcal{Z}(X_{[1,n]})$, and the zero element in $\mathcal{Z}(U_{[1,n]})$. Similarly for any $I \subset [1, n]$ we have an element $\Delta(I) \in \mathcal{F}(I, \emptyset)$; it is a cocycle of degree zero. As an element of $\mathcal{F}(I)$, it has degree 1.

For a subset $\Sigma \subset \overset{\circ}{I}$, letting $I_1, \cdots, I_c$ be the segmentation of I given by Σ, one verifies the tensor product

$$\Delta(I|\Sigma) := \Delta(I_1) \otimes \Delta(I_2) \otimes \cdots \otimes \Delta(I_c) \in \mathcal{F}(I_1, \emptyset) \otimes \cdots \otimes \mathcal{F}(I_c, \emptyset)$$

is indeed in the subcomplex $\mathcal{F}(I, \emptyset|\Sigma)$. As an element of the complex $\mathcal{F}(I|\Sigma)$, its degree is c. The elements $\Delta(I)$ are closed under ρ and π, namely:

(1) For $k \in \Sigma$, $\rho_k(\Delta(I|\Sigma)\,) = r_k(\Delta(I|\Sigma - \{k\})\,)$ in $\mathcal{F}(I, \{k\}|\Sigma - \{k\})$.

(2) For $K \subset \overset{\circ}{I} - \Sigma$, $\pi_K(\Delta(I|\Sigma)\,) = \Delta(I - K|\Sigma)$ in $\mathcal{F}(I - K|\Sigma)$.

By (1) one sees that the collection

$$\mathbf{\Delta}(I) := (\Delta(I|\Sigma)\,)_\Sigma \in \oplus\mathcal{F}(I, \emptyset|\Sigma) \subset F(I)$$

is a cocycle of degree 0 in the complex $F(I)$. If $I = [1, n]$, one should think of $\mathbf{\Delta}(I)$ as $\Delta([1, 2]) \otimes \cdots \otimes \Delta([n - 1, n])$, not as $\Delta([1, n])$. The following proposition contains a more precise statement.

(4.2) **Proposition.** *(1) If* $|I| = 2$, *then* $\mathbf{\Delta}(I) = \Delta(I) \in F(I)$.

(2) If $S \subset \overset{\circ}{I}$, *and* $I_1, \cdots, I_c$ *the corresponding segmentation, one has*

$$\tau_S(\mathbf{\Delta}(I)\,) = \mathbf{\Delta}(I_1) \otimes \cdots \otimes \mathbf{\Delta}(I_c)$$

in $F(I \top S) = F(I_1) \otimes \cdots \otimes F(I_c)$. *(Recall* $\tau_S : F(I) \to F(I \top S)$ *is the composition of* $\sigma_S : F(I) \to F(I|S)$ *and* $\iota_S : F(I|S) \to F(I \top S)$.*)*

(3) For $K \subset \overset{\circ}{I}$, $\varphi_K(\mathbf{\Delta}(I)\,) = \mathbf{\Delta}(I - K)$.

Since $\mathbf{\Delta}(I)$ depends on X, we will write $\Delta_X(I)$ for $\Delta(I)$ and $\mathbf{\Delta}_X(I)$ for $\mathbf{\Delta}(I)$. If $|I| = 2$, $\Delta_X(I)$ is the usual diagonal Δ_X.

(4.3) *The diagonal embedding* δ_*. Let X be a sequence of varieties on $I = [1, n]$. Given an element $k \in I$ (we allow $k = 1$ or $k = n$) and an integer $m \geq 2$, let $I^\sim = [1, n]^\sim$ be the ordered set

$$\{1, \cdots, k - 1, k_1, \cdots, k_m, k + 1, \cdots, n\} \ ,$$

where k is repeated m times. There is a natural surjection $I^{\sim} \to I$ which sends k_j to k and is the identity on $I^{\sim} - \{k_j\}$, so there is an induced sequence of varieties on $I^{\sim}$. Let X_I and $X_{I^{\sim}}$ be the corresponding varieties. There is a closed embedding $\delta : X_I \to X_{I^{\sim}}$ given by $(x_1, \cdots, x_n) \mapsto (x_1, \cdots, x_{k-1}, x_k, \cdots, x_k, x_{k+1}, \cdots, x_n)$ (x_k repeated m times). Note all this makes sense for any subset $I \subset [1, n]$, an element $k \in I$, and $m \geq 2$.

For the statement of the following proposition only, we write $\mathbb{I}$ (resp. $\mathbb{I}^{\sim}$) instead of I (resp. $I^{\sim}$). Recall for a subset $I \subset \mathbb{I}$ there corresponds a closed set $A_I \subset X_{\mathbb{I}}$, and U_I is its complement. Thus for $I' \subset \mathbb{I}^{\sim}$ the corresponding set is $A_{I'} \subset X_{\mathbb{I}^{\sim}}$. One verifies:

Proposition. *(1) Let $I \subset \mathbb{I}$ and $I' \subset \mathbb{I}^{\sim}$ be subsets such that $I' - \{k_j\} \xrightarrow{\sim} I - \{k\}$ and $I' \to I$ is a surjection. Then the following square is Cartesian:*

$$\begin{tikzcd} A_I \arrow[r, hook] \arrow[d] & X_{\mathbb{I}} \arrow[d, "\delta"] \\ A_{I'} \arrow[r, hook] & X_{\mathbb{I}^{\sim}} . \end{tikzcd}$$

Hence $\delta^{-1}(U_{I'}) = U_I$.

(2) If $\mathcal{J} \subset \overset{\circ}{\mathbb{I}}$ and $\mathcal{J}' \subset (\mathbb{I}^{\sim})^{\circ}$ are subsets such that $\mathcal{J}' \xrightarrow{\sim} \mathcal{J}$, then $\delta^{-1}\mathcal{U}(\mathcal{J}') = \mathcal{U}(\mathcal{J})$. We thus have a map of complexes (see (1.3))

$$\delta_* : \mathcal{Z}(X_{\mathbb{I}}, \mathcal{U}(\mathcal{J})\,) \to \mathcal{Z}(X_{\mathbb{I}^{\sim}}, \mathcal{U}(\mathcal{J}')\,)\ .$$

We refer to this $\delta_* : \mathcal{F}(I, \mathcal{J}) \to \mathcal{F}(I^{\sim}, \mathcal{J}')$ as the diagonal embedding *associated to the surjection $\mathbb{I}^{\sim} \to \mathbb{I}$.*

Proof. (1) Left to the reader.

(2) If $k \notin \mathcal{J}$, let $\{J^0, \cdots, J^r\}$ be the segmentation of $\mathbb{I}$ by $\mathcal{J}$. There is i such that $k \in J^i$. Then the segmentation of $I^{\sim}$ by $\mathcal{J}'$ is $\{\tilde{J}^0, \cdots, \tilde{J}^r\}$, where $\tilde{J}^j$ is the inverse image of J^j; $\tilde{J}^j$ is bijective to J^j if $j \neq i$. Apply (1) to $\tilde{J}^j$ and J^j for each j to obtain the claim. The case $k \in \mathcal{J}$ is similar.

(4.4) *The maps δ_* and $\Delta(\Sigma, \Sigma')$.* Keeping the notation, we will define a map of complexes

$$\mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I^{\sim}, \mathcal{J}'|\Sigma')$$

when the following condition is satisfied:

$$\mathcal{J}' \xrightarrow{\sim} \mathcal{J},\ \Sigma' - \{k_j\} \xrightarrow{\sim} \Sigma - \{k\}, \text{ and, if } k \in \Sigma \text{ then } \Sigma' \cap \{k_1, \cdots, k_m\} \text{ is non-empty.}$$

(If $k \notin \Sigma$, $\Sigma' \cap \{k_1, \cdots, k_m\}$ may be empty.) According to cases, we will give it the name δ_* or $\Delta(\Sigma, \Sigma')$. From now on we assume $k \neq 1, n$; at the end of this subsection we will mention the necessary changes in the case $k = 1$ or n.

(0) Case $k \notin \Sigma$. If Σ is the empty set, we have the map $\delta_* : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(I^{\sim}, \mathcal{J}'|\Sigma')$ defined in the previous subsection. There are two subcases:

(a) Case $k \notin \mathcal{J}$. Then $\mathcal{J}'$ as above is uniquely determined.

(b) Case $k \in \mathcal{J}$. Then $\mathcal{J}' = (\mathcal{J} - \{k\}) \cup \{k_j\}$ for $j = 1, \cdots, m$. So we write $(\delta_j)_*$ for δ_*.

One shows:

(4.4.1) **Lemma.** *(1) In cases (a) and (b), δ_* commutes with $r_{k'}$, $k' \neq k$. For δ_* in (a), the following commutes:*

$$\begin{tikzcd}
\mathcal{F}(I, \mathcal{J}) \arrow[r, "\delta_*"] \arrow[d, "r_k"'] & \mathcal{F}(\tilde{I}, \mathcal{J}') \arrow[d, "r_{k_j}"] \\
\mathcal{F}(I, \mathcal{J} \cup \{k\}) \arrow[r, "(\delta_j)_*"] & \mathcal{F}(\tilde{I}, \mathcal{J}' \cup \{k_j\})\ .
\end{tikzcd}$$

(2) In case (b), let $\mathcal{J} = \mathcal{J}_0 \cup \{k\}$. If $k \notin \mathcal{J}$ and $j \neq j'$, the following commutes:

$$\begin{tikzcd}
\mathcal{F}(I, \mathcal{J}) \arrow[r, "(\delta_j)_*"] \arrow[d, "(\delta_{j'})_*"'] & \mathcal{F}(\tilde{I}, \mathcal{J}_0 \cup \{k_j\}) \arrow[d, "r_{k_{j'}}"] \\
\mathcal{F}(I, \mathcal{J}_0 \cup \{k_{j'}\}) \arrow[r, "r_{k_j}"] & \mathcal{F}(\tilde{I}, \mathcal{J}_0 \cup \{k_j, k_{j'}\})\ .
\end{tikzcd}$$

Proof. (1) is left to the reader. The point in the proof of (2) is, if $\delta : U_{k,\cdots,n} \hookrightarrow U_{k_j,\cdots,k_m,k+1,\cdots,n}$ denotes the diagonal embedding, its image is disjoint from the subset $U_{k_j,\cdots,k_{j'}}$, $k_j < k_{j'}$.

For each $c \geq 0$ consider the direct sum $\bigoplus_{|\mathcal{J}|=c} \mathcal{F}(I, \mathcal{J})$, where $\mathcal{J} \subset \mathring{I}$ varies over subsets with cardinality c, and similarly $\bigoplus_{|\mathcal{J}'|=c} \mathcal{F}(\tilde{I}, \mathcal{J}')$. Let $\sum \delta_* : \bigoplus_{|\mathcal{J}|=c} \mathcal{F}(I, \mathcal{J}) \to \bigoplus_{|\mathcal{J}'|=c} \mathcal{F}(\tilde{I}, \mathcal{J}')$ be the sum of all δ_* defined above. The lemma implies that it commutes with r (the signed sum of r_i), so it gives a map of complexes $\mathcal{F}(I) \to \mathcal{F}(\tilde{I})$.

If Σ is not empty, but does not contain k, one generalizes the above in the obvious way and defines the map $\delta_* : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma')$. The above lemma also generalizes, so the sum of δ_* commutes with r.

Assume now $k \in \Sigma$, $\Sigma' \subset (\tilde{I})^\circ$ such that $\Sigma' - \{k_1, \cdots, k_m\} \xrightarrow{\sim} \Sigma - \{k\}$ and $\Sigma' \twoheadrightarrow \Sigma$. Let $\mathcal{J}' \subset (\tilde{I})^\circ$ be a subset such that $\mathcal{J}' \xrightarrow{\sim} \mathcal{J}$; since $k \notin \mathcal{J}$, $\mathcal{J}'$ is uniquely determined. We have two cases:

(I) Case $k \in \Sigma$ and $|\Sigma'| = |\Sigma|$. One can define $\Delta(\Sigma, \Sigma') : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma')$. For simplicity assume $\Sigma = \{k\}$, and let I_1, I_2 be the segmentation of I by k. Let $\ell = k_j$ be the element in Σ', I_1', I_2' be the segmentation of $\tilde{I}$ by ℓ, and δ', δ'' be the embeddings corresponding to the surjections $I_i' \to I_i$. Then the map $\Delta(\Sigma, \Sigma') : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma')$ is defined by $\Delta(\Sigma, \Sigma')(u' \otimes u'') = \delta'_*(u') \otimes \delta''_*(u'')$. That this definition makes sense follows from the following claim.

Claim. Let u_i be elements in $\mathcal{Z}(X_{I_i})$ for $i = 1, 2$, such that $\{u_1, u_2, faces\}$ is properly intersecting in X_I (so one has $u_1 \circ u_2 \in \mathcal{Z}(X_I)$ defined). Then for the cycles $\delta'_*(u_1), \delta''_*(u_2)$, respectively on $X_{I_i'}$, $i = 1, 2$, the set $\{\delta'_*(u_1), \delta''_*(u_2), faces\}$ is properly intersecting in $X_{\tilde{I}}$, and one has

$$\delta_*(u_1 \circ u_2) = \delta'_*(u_1) \circ \delta''_*(u_2)$$

in $\mathcal{Z}(X_{\tilde{I}})$.

(4.4.2) **Lemma.** *Assume we are in case (I); let $\Sigma' = (\Sigma - \{k\}) \cup \{k_j\}$.*

(1) $\Delta(\Sigma, \Sigma')$ commutes with $r_{k'}$ if $k' \in \mathring{I} - (\mathcal{J} \cup \Sigma)$.

(2) $\Delta(\Sigma,\Sigma')$ commutes with $\rho_{k'}$ if $k'\neq k$. Further, the following square commutes:

$$\begin{array}{ccc}
\mathcal{F}(I,\mathcal{J}|\Sigma) & \xrightarrow{\Delta(\Sigma,\Sigma')} & \mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma') \\
\rho_k\big\downarrow & & \big\downarrow\rho_{k_j} \\
\mathcal{F}(I,\mathcal{J}\cup\{k\}|\Sigma-\{k\}) & \xrightarrow{(\delta_j)_*} & \mathcal{F}(I^{\sim},\mathcal{J}'\cup\{k_j\}|\Sigma'-\{k_j\})\ .
\end{array}$$

The assertion (2) follows from the identity $\delta_*(u_1\circ u_2)=\delta'_*(u_1)\circ\delta''_*(u_2)$ in the claim.

(II) Case $|\Sigma'|>|\Sigma|$. We will define the map

$$\Delta(\Sigma,\Sigma'):\mathcal{F}(I,\mathcal{J}|\Sigma)\to\mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma')$$

as follows. For simplicity assume $\Sigma=\{k\}$, the general case being similar. Let I_1,I_2 be the segmentation of I by k, and $I'_1,\cdots,I'_{b+1}$ the segmentation of $I^{\sim}$ by Σ'. One has $\mathcal{F}(I,\mathcal{J}|\Sigma)=\mathcal{F}(I_1,\mathcal{J}_1)\hat{\otimes}\mathcal{F}(I_2,\mathcal{J}_2)$, and

$$\mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma')=\mathcal{F}(I'_1,\mathcal{J}'_1)\hat{\otimes}\mathcal{F}(I'_2,\emptyset)\hat{\otimes}\cdots\hat{\otimes}\mathcal{F}(I'_b,\emptyset)\hat{\otimes}\mathcal{F}(I'_{b+1},\mathcal{J}'_{b+1})\ .$$

Note $I'_2,\cdots,I'_b$ correspond to constant sequences on X_k. The map $\Delta(\Sigma,\Sigma')$ is defined by

$$u'\otimes u''\mapsto\delta'_*(u')\otimes\Delta(I'_2)\otimes\cdots\otimes\Delta(I'_b)\otimes\delta''_*(u'')$$

where $\delta'_*:\mathcal{F}(I_1,\mathcal{J}_1)\to\mathcal{F}(I'_1,\mathcal{J}'_1)$ is the map associated to the surjection $I'_1\to I_1$, and similarly for the map δ''_*. We have used the following claim.

Claim. Let u_i be elements in $\mathcal{Z}(X_{I_i})$ for $i=1,2$, such that $\{u_1,u_2,faces\}$ is properly intersecting in X_I (so one has $u_1\circ u_2\in\mathcal{Z}(X_I)$ defined). Then the set of cycles

$$\{\delta'_*(u_1),\Delta(I'_2),\cdots,\Delta(I'_b),\delta''_*(u_2),faces\}$$

is properly intersecting in $X_{I^{\sim}}$. One has

$$\delta'_*(u_1)\circ\Delta(I'_2)=\bar{\delta}'_*(u_1)$$

where $\bar{\delta}'$ is associated to the surjection $I'_1\cup I'_2\to I_1$; similarly for $\Delta(I'_b)\circ\delta''_*(u_2)$.

(4.4.3) **Lemma.** *Assume we are in case (II).*
(1) $\Delta(\Sigma,\Sigma')$ commutes with $r_{k'}$ if $k'\neq k$, and with $\rho_{k'}$ if $k'\neq k$.
(2) If $\ell=k_j\in\Sigma'$, the following commutes:

$$\begin{array}{ccc}
\mathcal{F}(I,\mathcal{J}|\Sigma) & \xrightarrow{\Delta(\Sigma,\Sigma')} & \mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma') \\
\Delta(\Sigma,\Sigma'-\{\ell\})\big\downarrow & & \big\downarrow\rho_\ell \\
\mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma'-\{\ell\}) & \xrightarrow{r_\ell} & \mathcal{F}(I^{\sim},\mathcal{J}'\cup\{\ell\}|\Sigma'-\{\ell\})\ .
\end{array}$$

(4.4.4) *Case $k=1$ or n.* If $k=n$, minor changes are needed as follows.
(a) In case $\Sigma'\cap\{n_1,\cdots,n_m\}=\emptyset$ we have the map $\delta_*:\mathcal{F}(I,\mathcal{J}|\Sigma)\to\mathcal{F}(I^{\sim},\mathcal{J}'|\Sigma')$. This is defined as in case (0) above. Lemma (4.4.1) holds without change.

(b) In case $\Sigma' \cap \{n_1, \cdots, n_m\} \neq \emptyset$, we have $\Delta(\Sigma, \Sigma') : \mathcal{F}(I, \mathcal{J}|\Sigma) \to \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma')$, defined as in case (II) above, by the formula $u \mapsto \delta'_*(u) \otimes \Delta \otimes \cdots \otimes \Delta$. Lemma (4.4.3) holds, where if Σ' consists of a single element one replaces $\Delta(\Sigma, \Sigma' - \{\ell\})$ by δ'_*.

(4.5) Consider now the map

$$\mathrm{diag} = \mathrm{diag}(I, \tilde{I}) = \sum \delta_* + \sum \Delta(\Sigma, \Sigma') : \bigoplus \mathcal{F}(I, \mathcal{J}|\Sigma) \to \bigoplus \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma')$$

which is the sum of δ_* and $\Delta(\Sigma, \Sigma')$. The three lemmas jointly imply:

Proposition. *The map* diag *commutes with* $\bar{r} + \bar{\rho}$.

Proof. Assume $k \neq 1, n$ (the proof is similar in those cases). By the lemmas, we have:

$$r(\sum \delta_*) = (\sum \delta_*) r \ ;$$

For $\Delta(\Sigma, \Sigma')$ of type (I) or (II), $k' \neq k_j$,

$$r_{k'} \Delta(\Sigma, \Sigma') = \Delta(\Sigma, \Sigma') r_{k'} \ ,$$

$$\rho_{k'} \Delta(\Sigma, \Sigma') = \Delta(\Sigma, \Sigma') \rho_{k'} \ ;$$

For $\Delta(\Sigma, \Sigma')$ of type (I),

$$\rho_{k_j} \Delta(\Sigma, \Sigma') = (\delta_j)_* \rho_k \ ;$$

Also,

$$\sum_{\mathrm{type(II)}} \sum_{\substack{k_j \in \Sigma' \\ \mathrm{over} k}} \rho_{k_j} \Delta(\Sigma, \Sigma') = \sum_{\mathrm{type(I) or (II)}} \sum_{k_j \in \Sigma'} r_{k_j} \Delta(\Sigma, \Sigma') \ .$$

In calculating $(\bar{r}+\bar{\rho})\,\mathrm{diag}$, in light of the last identity one can disregard the terms $\sum \rho_{k_j} \Delta(\Sigma, \Sigma')$, the sum over type (II), and $\sum r_{k_j} \Delta(\Sigma, \Sigma')$, the sum over type (I) or (II). For the other identities above, careful examination of the signs show that they still hold if $r_{k'}$ (resp. $\rho_{k'}$) is replaced by $\bar{r}_{k'}$ (resp. $\bar{\rho}_{k'}$). Hence we obtain the assertion.

(4.6) The map $\mathrm{diag} : F(I) \to F(\tilde{I})$ is compatible with φ and τ:

Proposition. *(1) If $k' \neq k$, $\varphi_{k'}\,\mathrm{diag}(I, \tilde{I}) = \mathrm{diag}(I - \{k'\}, \tilde{I} - \{k'\}) \varphi_{k'}$, namely the following square commutes:*

$$\begin{array}{ccc} F(I) & \xrightarrow{\mathrm{diag}(I, \tilde{I})} & F(\tilde{I}) \\ {\scriptstyle \varphi_{k'}} \downarrow & & \downarrow {\scriptstyle \varphi_{k'}} \\ F(I - \{k\}) & \xrightarrow{\mathrm{diag}(I - \{k'\}, \tilde{I} - \{k'\})} & F(\tilde{I} - \{k'\}) \ . \end{array}$$

If $\ell \in \{k_1, \cdots, k_m\}$, $\varphi_\ell\,\mathrm{diag}(I, \tilde{I}) = \mathrm{diag}(I, \tilde{I} - \{\ell\})$; if $m = 2$ interpret the right hand side as the identity.

(2) If $k = n$, $\ell \in \{n_1, \cdots, n_m\}$, let I'_1, I'' be the segmentation of $\tilde{I}$ by ℓ. Then the following diagram commutes:

$$\begin{array}{ccc} F(I) & \xrightarrow{\mathrm{diag}(I, \tilde{I})} & F(\tilde{I}) \\ {\scriptstyle \mathrm{diag}(I, I'_1)} \downarrow & & \downarrow {\scriptstyle \tau_\ell} \\ F(I'_1) & \longrightarrow & F(I'_1) \otimes F(I'') \ . \end{array}$$

The lower horizontal map is $u \mapsto u \otimes \mathbf{\Delta}(I'')$. *Note* I'' *parametrizes a constant sequence of varieties, so one has* $\mathbf{\Delta}(I'') \in F(I'')$. *Similarly in case* $k = 1$, $\ell \in \{1_1, \cdots, 1_m\}$.

If $1 < k < n$ *and* $\ell \in \{k_1, \cdots, k_m\}$, *let* I_1, I_2 *be the segmentation of* I *by* k, *and* I_1', I_2' *of* $\tilde{I}$ *by* ℓ. *One then has a commutative diagram:*

$$\begin{tikzcd} F(I) \arrow[r, "\mathrm{diag}(I,\tilde{I})"] \arrow[d, "\tau_k"'] & F(\tilde{I}) \arrow[d, "\tau_\ell"] \\ F(I_1) \otimes F(I_2) \arrow[r] & F(I_1') \otimes F(I_2') \end{tikzcd},$$

where the lower horizontal arrow is $\mathrm{diag}(I_1, I_1') \otimes \mathrm{diag}(I_2, I_2')$.

Proof. We only verify the last statement. The map $\Delta(\Sigma, \Sigma')$ is defined so that if $\ell \in \Sigma'$, the following commutes:

$$\begin{tikzcd} \mathcal{F}(I, \mathcal{J}|\Sigma) \arrow[r, "\Delta(\Sigma,\Sigma')"] \arrow[d] & \mathcal{F}(\tilde{I}, \mathcal{J}'|\Sigma') \arrow[d] \\ \mathcal{F}(I_1, \mathcal{J}_1|\Sigma_1) \otimes \mathcal{F}(I_2, \mathcal{J}_2|\Sigma_2) \arrow[r] & \mathcal{F}(I_1', \mathcal{J}_1'|\Sigma_1') \otimes \mathcal{F}(I_2', \mathcal{J}_2'|\Sigma_2') \end{tikzcd}.$$

Here $\mathcal{J}_i = \mathcal{J} \cap \overset{\circ}{I}_i$, $\Sigma_i = \Sigma \cap \overset{\circ}{I}_i$, and similarly for $\mathcal{J}_i'$ and Σ_i'. The vertical inclusions are the canonical ones, and the lower horizontal arrow is $\Delta(\Sigma_1, \Sigma_1') \otimes \Delta(\Sigma_2, \Sigma_2')$. Taking the sum over $\Delta(\Sigma, \Sigma')$ we obtain the claim.

(4.7) All of (4.3)-(4.6) can be extended as follows. Given a subset $\{k, k', k'', \cdots\}$ of $I = [1, n]$, and a set of integers ≥ 2, $m, m', m'', \cdots$, let

$$\tilde{I} = \{1, \cdots, k-1, k_1, \cdots, k_m, \cdots, k_1', \cdots, k_{m'}', \cdots, n\}$$

be the ordered set where $k, k', k'', \cdots$ are repeated $m, m', m'', \cdots$ times. One can then define the diagonal extension $\mathrm{diag} : F(I) \to F(\tilde{I})$ that satisfies properties as above.

(4.8) One can state more generally assumptions on a set of complexes $A(I, \mathcal{J}|\Sigma)$ satisfying Assumption (A) in §2, under which the same constructions can be performed.

For a constant sequence $I \ni i \mapsto X$, we assume, as in (4.1), the existence of a distinguished element $\Delta(I) \in A(I, \emptyset)$, which is a cocycle of degree 0. Require that the tensor products $\Delta(I|\Sigma)$ are in $A(I, \emptyset|\Sigma)$, and they are subject to the same identities with respect to ρ, r, π as in (4.1). Then the element $\mathbf{\Delta}(I) \in A(I)$ is defined, and (4.2) satisfied, with $F(I|S)$ replaced with $B(I|S)$.

Also assume there are maps of complexes $\delta_* : A(I, \mathcal{J}|\Sigma) \to A(\tilde{I}, \mathcal{J}'|\Sigma')$ when $k \notin \Sigma$, and require Lemma (4.4.1) to hold. When $k \in \Sigma$, assume there are maps $\Delta(\Sigma, \Sigma') : A(I, \mathcal{J}|\Sigma) \to A(\tilde{I}, \mathcal{J}'|\Sigma')$, that are defined using δ_* and tensor product as in (4.4), for which (4.4.2) and (4.4.3) hold.

Under these assumptions one can define the the map $\mathrm{diag} : B(I) \to B(\tilde{I})$ and Proposition (4.6) is satisfied.

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