Homological Hodge complexes III

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Consolidating Part I and Part II, we will construct the homological Hodge complex for a pair of smooth varieties. More precisely, we take a smooth complete variety X, a pair of normal crossing divisors Y and H on X, consider the pair $(X - Y, H - Y)$, and construct a Hodge complex for the pair. In Part I we considered the case Y empty, and in Part II the case H empty.

1 Preliminaries

In this section we will consider complexes of vector spaces over $\mathbb{C}$ to fix the idea; nothing changes if we take any other coefficient field.

(1.1) For a complex K, let $D(K)$ be the complex with $\mathrm{Hom}(K^{-i}, \mathbb{C})$ as the degree i part, and with differential

$$f \mapsto f \circ d\,.$$

Let K and L be complexes of $\mathbb{C}$-vector spaces. A map of complexes $\langle\ ,\ \rangle : K \otimes L \to \mathbb{C}$ is the same as a graded map such that for any $x \in K^i$ and $y \in L^{-i-1}$, one has

$$\langle dx, y\rangle + (-1)^i \langle x, dy\rangle = 0\,.$$

Such a map is said to be a pairing between between K and L.

There is an isomorphism of complexes $\sigma : K \otimes L \to L \otimes K$ which sends $x \otimes y \in K^i \otimes L^j$ to $(-1)^{ij} y \otimes x \in L^j \otimes K^i$. One obtains a pairing $\langle\ ,\ \rangle' : L \otimes K \to \mathbb{C}$ which commutes with the original pairing via this isomorphism.

A pairing induces a map of complexes $\mathcal{P} : K \to D(L)$, given for $x \in K^i$ and $y \in L^{-i}$ by

$$\mathcal{P}(x)(y) = (-1)^{s(i)} \langle x, y\rangle, \quad s(i) = \frac{i(i+1)}{2}\,.$$

(1.2) Suppose $K \otimes L \to \mathbb{C}$ and $K' \otimes L' \to \mathbb{C}$ are pairings of complexes, and $u : K \to K'$, $v : L' \to L$ are maps of complexes.

We say that the pairings are compatible with respect to the maps u and v, if the identity $\langle u(x), y' \rangle = \langle x, v(y') \rangle$ holds for $x \in K$ and $y' \in L'$, namely if the diagram

$$\begin{tikzcd} K \otimes L \arrow[r, "\langle\,,\rangle"] \arrow[d, "u"'] & \mathbb{C} \arrow[d, equal] \\ K' \otimes L' \arrow[u, "v"'] \arrow[r, "\langle\,,\rangle"'] & \mathbb{C} \end{tikzcd} \tag{1.2.1}$$

commutes. If $\mathcal{P} : K \to D(L)$ and $\mathcal{P}' : K' \to D(L')$ are the maps associated with the pairings, and $v' = D(v) : D(L) \to D(L')$ is the map induced from v, then the diagram

$$\begin{tikzcd} K \arrow[r, "\mathcal{P}"] \arrow[d, "u"'] & D(L) \arrow[d, "v'"] \\ K' \arrow[r, "\mathcal{P}'"'] & D(L') \end{tikzcd}$$

commutes.

Instead of commutativity on (1.2.1), suppose we have a map $h : K \otimes L' \to \mathbb{C}$ of degree -1 such that

$$h(d(x \otimes y')) = \langle u(x), y' \rangle - \langle x, v(y') \rangle \quad \text{for} \quad x \in K^i,\, y' \in L'^{-i+1}\,.$$

Such a map is said to be a *homotopy* between the two pairings. Consider then a map of degree -1, $H : K \to D(L')$ given for $x \in K^i$, $y' \in L'^{-i+1}$ by

$$H(x)(y') = (-1)^{i(i+1)/2} h(x \otimes y').$$

One has the identity $d \circ H + H \circ d = \mathcal{P}' \circ u - v' \circ \mathcal{P}$.

(1.3) Let

$$\longrightarrow K^a \xrightarrow{u} K^{a+1} \xrightarrow{u} \cdots$$

be a finite compex of complexes. This means that given a sequence of complexes $K^a = (K^{a,\bullet}, d)$, with K^a being the zero complex for all but finitely many a, and that given a map of complexes $u : K^a \to K^{a+1}$ for each a satisfying $u \circ u = 0$.

The total complex of this, denoted by $\mathrm{Tot}(K^\bullet)$ is a complex with degree i part

$$(\mathrm{Tot}(K^\bullet))^i = \bigoplus_a K^{a, i-a}\,,$$

and with differential $(-1)^a d_{K^a} + u$ on K^a.

If $(K^a; u)$ is a complex of complexes, then the the dual of $u : K^a \to K^{a+1}$ is a map of complexes $D(K^{a+1}) \xrightarrow{u'} D(K^a)$. So there is a complex of complexes

$$\cdots \to D(K^{a+1}) \xrightarrow{u'} D(K^a) \xrightarrow{u'} D(K^{a-1}) \to \cdots$$

with $D(K^a)$ in horizontal degree $-a$.

The identity map gives an isomorphism of complexes

$$D(\mathrm{Tot}(K^\bullet)) = \mathrm{Tot}(D(K^\bullet))\,. \tag{1.3.1}$$

(1.4) Suppose

$$\cdots \xleftarrow{v} L_a \xleftarrow{v} L_{a+1} \leftarrow \cdots$$

is another finite complex of complexes, and assume given pairings $\langle\,,\,\rangle_a : K_a \otimes L_a \to \mathbb{C}$ such that the pairings $\langle\,,\,\rangle_a$ and $\langle\,,\,\rangle_{a+1}$ are compatible via the maps u, v, meaning that the identity

$$\langle u(x_a), y_{a+1}\rangle_a = \langle x_a, v(y_{a+1})\rangle_a$$

holds for $x_a \in K^a$ and $y_{a+1} \in L_{a+1}$ of the same degree.

Then one can define a pairing of the total complexes

$$\langle\,,\,\rangle : \mathrm{Tot}(K) \otimes \mathrm{Tot}(L) \to \mathbb{C}$$

by the formula

$$\langle (x_a), (y_a)\rangle = \sum_a (-1)^{\gamma(i,a)} \langle x_a, y_a\rangle_a\,, \quad \gamma(i,a) = ia + s(a-1) \tag{1.4.1}$$

for $(x_a) \in \bigoplus K^{a,i-a}$ and $(y_a) \in \bigoplus L_a^{-i+a}$. The reader may verify that this gives a pairing.

For each a the pairing $\langle\,,\,\rangle_a$ induces a map of complexes

$$\mathcal{P}_a : K^a \to D(L_a)$$

according to (1.1), which form a morphism between the complexes of complexes

$$K_\bullet \to D(L_\bullet)\,.$$

Taking the total complexes, one has

$$\mathrm{Tot}(K_\bullet) \to \mathrm{Tot}(D(L_\bullet)\,)\,;$$

composing with the identification $D(\mathrm{Tot}(L_\bullet)) \to \mathrm{Tot}(D(L_\bullet))$ one may view this as a map $\mathrm{Tot}(K_\bullet) \to D(\mathrm{Tot}(L_\bullet))$. It can be verified that this coincides with the map induced from the pairing (1.4.1) according to (1.1).

(1.5) The considerations for complexes of complexes in the preceding subsections extend to double complexes of complexes. Suppose $(K^{a,b}; u', u'')$ is a finite double complex of complexes. This means each $K^{a,b}$ is a complex, $u' : K^{a,b} \to K^{a+1,b}$ (horizontal differential) and $u'' : K^{a,b} \to K^{a,b+1}$ (vertical differential) are maps of complexes that satisfy the identities

$$u'u' = 0, \quad u''u'' = 0, \quad u'u'' = u''u'\,.$$

We can then define its total complex Tot(K) by

$$(\mathrm{Tot}(K^\bullet))^i = \bigoplus K^{a,b,i-a-b}$$

as a graded group, together with differential (write d for $d_{K^{a,b}}$)

$$(-1)^{a+b}d + (-1)^a u' + u''$$

on $K^{a,b}$.

One may view the double complex $K^{a,b}$ as a complex of complexes by totalizing in the vertical differential u'' and d,

$$\longrightarrow \bigoplus_b K^{a,b} \xrightarrow{u'} \bigoplus_b K^{a+1,b} \xrightarrow{u'} \cdots .$$

The totalization of this coincides with Tot(K) give above. One may likewise totalize in the differential u' and d to obtain a complex of complexes, and the totalize it; one obtains the same result.

Suppose given another double complex of complexes $(L_{a,b}, v', v'')$ with $v' : L_{a,b} \to L_{a-1,b}$ and $v'' : L_{a,b} \to L_{a,b-1}$, and pairings $K^{a,b} \otimes L_{a,b} \to \mathbb{C}$ that are compatible with respect to the maps u' and v', and u'' and v''. Then there is an induced map pairing of the total complexes.

2 Complexes of topological chains, forms and currents

We use the same notations as in Part I and II.

We will consider complexes of sheaves of Λ-vector spaces on a variety X, with $\Lambda = \mathbb{Q}$ or $\mathbb{C}$; they may be simply called *complexes on* X when there is no possibility of confusion.

(2.1) A pair (X, H) consisting of a smooth complete variety X and a simple normal crossing divisor H is called a smooth pair.

We always assume that the irreducible components of H are totally ordered, $H_1, \cdots, H_r$; we say $\{1, \cdots, r\}$ is the index set for H, and write it as $Ind(H)$ when necessary.

If I and J is a pair of the subsets $\{1, \cdots, r\}$ with $J \supset I$ and $|J| = |I| + 1$, we will write $J \rhd I$. For a subset I of $\{1, \cdots, r\}$, we set

$$H_I = \cap_{i \in I} H_i \,,$$

and $H_\emptyset = X$. H_I is a non-singular variety. Also let

$$\widehat{H_I} = \sum_J H_J \,,$$

where J varies over the subsets with $J \rhd I$; it is a normal crossing divisor on H_I. Thus a smooth pair $(H_I, \widehat{H_I})$ is reproduced.

For an $a \geq 0$, set

$$H^{(a)} = \coprod_{|I|=a} (H_I, \widehat{H_I})$$

the disjoint union of the smooth pairs $(H_I, \widehat{H_I})$; we will write H^a for $H^{(a)}$ when there is no confusion.

Suppose Y is another normal crossing divisor on X that meets H transversally; let $Y_1, \cdots, Y_s$ are the irreducible components. Then to a subset J of $\{1, \cdots, s\}$ there corresponds the subvariety Y_J. On the smooth variety $H_I \cap Y_J$, we have a pair of normal crossing divisors $\widehat{H_I} \cap Y_J$ and $H_I \cap \widehat{Y_J}$ which meet transversally.

Complexes of topological chains

(2.2) Let $\mathcal{C}(X) = \mathcal{C}(X)^\bullet$ be the complex of topological chains on X; it is concentrated in cohomological degree $[0, 2n]$.

Given a normal crossing divisor H on X, let $\mathcal{C}(X)_H = \mathcal{C}(X)^\bullet_H$ be the subcomplex of sheaves of $\mathcal{C}(X)^\bullet$ consisting of chains α that are admissible with respect to H. The inclusion $\mathcal{C}(X)_H \to \mathcal{C}(X)$ is a quasi-isomorphism.

For Z a closed smooth subvariety of X of codimension c, which meets H transversally, there is a map of complexes $\mathrm{in}_Z : \mathcal{C}(Z)_{Z \cap H}[-2c] \to \mathcal{C}(X)_H$ which takes an element α to itself.

Suppose H is a normal crossing divisor on X. For each H_I we have a map $\mathrm{rest}_I : \mathcal{C}(X)_H \to \mathcal{C}(H_I)_{\widehat{H_I}}$ which sends a chain α to $i^*(\alpha) = \alpha . H_I$; more generally for a pair with $I \subset J$, there is $\mathrm{rest}_{I,J} : \mathcal{C}(H_I)_{\widehat{H_I}} \to \mathcal{C}(H_J)_{\widehat{H_J}}$. One has transitivity: for subsets $I \subset J \subset K$ one has $\mathrm{rest}_{I,K} = \mathrm{rest}_{J,K}\,\mathrm{rest}_{I,J}$.

For $a \geq 0$ set $\mathcal{C}(H^{(a)})_{\widehat{H^{(a)}}} = \bigoplus_{|I|=a} \mathcal{C}(H_I)_{\widehat{H_I}}$, and define the map

$$i_H^* : \mathcal{C}(H^{(a)})_{\widehat{H^{(a)}}} \to \mathcal{C}(H^{(a+1)})_{\widehat{H^{(a+1)}}}$$

as the sum of the maps $\epsilon(I,J) \cdot \mathrm{rest}_{I,J}$ over the pairs (I,J) with $I \lhd J$ and I of order a. Here the sign $\epsilon = \pm 1$ is defined as follows: $I = (i_1, \cdots, i_a)$ in the increasing order and $J = (i_1, \cdots, i_k, j, i_{k+1}, \cdots i_a)$, let $\epsilon(I,J) = (-1)^k$.

One has $i_H^* i_H^* = 0$, so that there is a complex of complexes

$$0 \to \mathcal{C}(X)_H \xrightarrow{i_H^*} \mathcal{C}(H^{(1)})_{\widehat{H^{(1)}}} \longrightarrow \cdots \longrightarrow \mathcal{C}(H^{(a)})_{\widehat{H^{(a)}}} \xrightarrow{i_H^*} \cdots$$

with the term $\mathcal{C}(X)_H$ placed in degree 0. This and its total complex will be denoted by $\mathcal{C}(X|H)$.

For Y a normal crossing divisor, one has the inclusion map

$$\mathrm{in}_{I,J} : \mathcal{C}(Y_J)[-2|J|] \to \mathcal{C}(Y_I)[-2|I|]$$

for $I \subset J$. For $b \geq 0$ define the map $i_{Y*} : \mathcal{C}(Y^{(b+1)})[-2] \to \mathcal{C}(Y^{(b)})$ by

$$i_{Y*} = \sum \epsilon(I,J) \cdot \mathrm{in}_{I,J}$$

for $I \lhd J$ and I of order b. We have a complex of complexes

$$\longrightarrow \mathcal{C}(Y^{(b)})[-2b] \xrightarrow{i_{Y*}} \cdots \longrightarrow \mathcal{C}(Y^{(1)})[-2] \xrightarrow{i_{Y*}} \mathcal{C}(X) \to 0$$

(with $\mathcal{C}(X)$ in degree 0) which will be denoted $\mathcal{C}(X \backslash Y)$.

For a variant of this, suppose now H be another normal crossing divisor on X meeting Y transversally. On each Y_J we have the normal crossing divisor $Y_J \cap H$ and the corresponding complex $\mathcal{C}(Y_J)_{Y_J \cap H}$. The map $\mathrm{in}_{I,J}$ for $I \subset J$ in the previous paragraph restricts to a map

$$\mathrm{in}_{I,J} : \mathcal{C}(Y_J)_{Y_J \cap H}\,[-2|J|] \to \mathcal{C}(Y_I)_{Y_J \cap H}\,[-2|I|]\,.$$

As before we get maps $i_{Y*} : \mathcal{C}(Y^{(b+1)})_{\widehat{Y^{(b+1)}}}[-2] \to \mathcal{C}(Y^{(b)})_{\widehat{Y^{(b)}}}$ and the complex of complexes

$$\longrightarrow \mathcal{C}(Y^{(b)})_{Y^{(b)} \cap H}[-2b] \xrightarrow{i_{Y*}} \cdots \longrightarrow \mathcal{C}(Y^{(1)})_{Y^{(1)} \cap H}[-2] \xrightarrow{i_{Y*}} \mathcal{C}(X)_H \to 0\,.$$

This we denote as $\mathcal{C}(X \backslash Y)_H$. The inclusion into $\mathcal{C}(X \backslash Y)$ is a quasi-isomorphism.

On the smooth variety $Y_J \cap H_I$ there is a normal crossing divisor $Y_J \cap \widehat{H_I}$ and the complex $\mathcal{C}(Y_J \cap H_I)_{Y_J \cap \widehat{H_I}}$. If $J \subset J'$, $|J'| = |J|+1$ and $I \subset I'$, $|I'| = |I|+1$, then we have a commutative diagram

$$\begin{array}{ccc}
\mathcal{C}(Y_{J'} \cap H_{I'})_{Y_{J'} \cap \widehat{H_{I'}}}[-2] & \xrightarrow{\mathrm{in}_{J,J'}} & \mathcal{C}(Y_J \cap H_{I'})_{Y_J \cap \widehat{H_{I'}}} \\
{\scriptstyle \mathrm{rest}_{I,I'}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{rest}_{I,I'}} \\
\mathcal{C}(Y_{J'} \cap H_I)_{Y_{J'} \cap \widehat{H_I}}[-2] & \xrightarrow{\mathrm{in}_{J,J'}} & \mathcal{C}(Y_J \cap H_I)_{Y_J \cap \widehat{H_I}}
\end{array}$$

Define now, for $a, b \geq 0$,

$$\mathcal{C}(Y^{(b)} \cap H^{(a)})_{Y^{(b)} \cap \widehat{H_I}} = \bigoplus_{|J|=b, |I|=a} \mathcal{C}(Y_J \cap H_I)_{Y_J \cap \widehat{H_I}}[-2b] \,.$$

With the maps i_{Y*} and $i_H{}^*$ defined as the signed sums as before, we have $i_{Y*} i_{Y*} = 0$, $i_H{}^* i_H{}^* = 0$, and the diagrams

$$\begin{array}{ccc} \mathcal{C}(Y^{(b+1)} \cap H^{(a+1)})_{Y^{(b+1)} \cap \widehat{H^{(a+1)}}} & \xrightarrow{i_{Y*}} & \mathcal{C}(Y^{(b)} \cap H^{(a+1)})_{Y^{(b)} \cap \widehat{H^{(a+1)}}} \\ {\scriptstyle i_H^*}\uparrow & & {\scriptstyle i_H^*}\uparrow \\ \mathcal{C}(Y^{(b+1)} \cap H^{(a)})_{Y^{(b+1)} \cap \widehat{H^{(a)}}} & \xrightarrow{i_{Y*}} & \mathcal{C}(Y^{(b)} \cap H^{(a)})_{Y^{(b)} \cap \widehat{H^{(a)}}} \end{array}$$

commute. Hence we obtain a double complex of complexes with terms $\mathcal{C}(Y^{(b)} \cap H^{(a)})_{Y^{(b)} \cap \widehat{H^{(a)}}}$ in $(-b, a)$, $a, b \geq 0$, and the maps i_* and i^*. We denote this double complex, as well as its total complex, by $\mathcal{C}(X \backslash Y | H)$.

The 0-th row of $\mathcal{C}(X \backslash Y | H)$ coincides with the complex $\mathcal{C}(X \backslash Y)_H$ introduced before, and the 0-th column coincides with $\mathcal{C}(X | H)$.

Complexes of sheaves of forms

(2.3) *The complex $\mathcal{A}(X)_H$.* Let $\mathcal{A}_X$ be the complex of sheaves of smooth differential forms on X, is also denoted $\mathcal{A}(X)$. For a closed subset Z of X, $\mathcal{A}_X | Z$ denotes the restriction of $\mathcal{A}_X$ to Z, often viewed as a complex of sheaves on X. If Z is a smooth subvariety, there is the complex $\mathcal{A}_Z$ of forms on Z. The induced map $\mathcal{A}_X | Z \to \mathcal{A}_Z$ is a quasi-isomorphism of sheaves on Z.

For each I there is a map of complexes, called the Poincaré residue map

$$R_{H_I} = R_I : \mathcal{A}(X)\langle H \rangle \to \mathcal{A}(H_I)\langle \widehat{H_I} \rangle[-|I|] \,,$$

with $I \subset J$ one has the map

$$R_{I,J} : \mathcal{A}(H_I)\langle \widehat{H_I} \rangle[-|I|] \to \mathcal{A}(H_J)\langle \widehat{H_J} \rangle[-|J|] \,;$$

recall the change of signs of the differential when a complex is shifted. To be precise, $R_{I,J}$ is a map of complexes of sheaves on H_I, where the target complex is identified with its direct image under the inclusion $i : H_J \to H_I$.

If $J = I \cup \{j\}$, one may write R_j for $R_{I,J}$. We have identities (shifts are omitted)

$$R_I = R_{i_a} \cdots R_{i_1} \quad \text{if } I = (i_1, \cdots, i_a),$$

$$R_i R_j = -R_j R_i : \mathcal{A}(H_I)\langle \widehat{H_I} \rangle \to \mathcal{A}(H_K)\langle \widehat{H_K} \rangle$$

if $K = I \cup \{i, j\}$, and

$$R_{I,J} R_I = (-1)^{a+k} R_J : \mathcal{A}(X)\langle H \rangle \to \mathcal{A}(H_J)\langle \widehat{H_J} \rangle$$

if $I = (i_1, \cdots, i_a)$ and $J = (i_1, \cdots, i_k, j, i_{k+1}, \cdots, i_a)$.

For $a \geq 0$ consider the sum $\bigoplus_{|I|=a} \mathcal{A}(H_I)\langle\widehat{H_I}\rangle$ which is a complex on X, and let

$$r : \bigoplus_{|I|=a} \mathcal{A}(H_I)\langle\widehat{H_I}\rangle[-|I|] \to \bigoplus_{|J|=a+1} \mathcal{A}(H_J)\langle\widehat{H_J}\rangle[-|J|]$$

be the sum of the maps $R_{I,J}$. Then $r \circ r = 0$, so that we have a complex of complexes (on X)

$$0 \to \mathcal{A}(X)\langle H\rangle \xrightarrow{r} \bigoplus_i \mathcal{A}(H_i)\langle\widehat{H_i}\rangle[-1] \xrightarrow{r} \bigoplus_{|I|=2} \mathcal{A}(H_I)\langle\widehat{H_I}\rangle[-2] \to \cdots$$

(the term $\mathcal{A}(X)\langle H\rangle$ is placed in degree 0).

Noting the identity

$$\mathcal{A}(H^{(a)})\langle\widehat{H^{(a)}}\rangle = \bigoplus_{|I|=a} \mathcal{A}(H_I)\langle\widehat{H_I}\rangle\,,$$

the above may be written

$$0 \to \mathcal{A}(X)\langle H\rangle \xrightarrow{r} \mathcal{A}(H^{(1)})\langle\widehat{H^{(1)}}\rangle[-1] \xrightarrow{r} \mathcal{A}(H^{(2)})\langle\widehat{H^{(2)}}\rangle[-2] \xrightarrow{r} \cdots\,.$$

This double complex, as well as its total complex, will be denoted $\mathcal{A}(X)_H$.

The differential of the total complex, restricted to $\mathcal{A}(H^{(a)})\langle\widehat{H^{(a)}}\rangle$, equals the sum of $(-1)^a$ times the differential of $\mathcal{A}(H^{(a)})\langle\widehat{H^{(a)}}\rangle[-a]$ and r, see (1.3). Recall also that shift of a complex changes the sign of the differential. Thus a section of $\mathcal{A}(X)_H$ of degree p is of the form

$$\varphi = (\varphi_I)_I, \quad \text{with } \varphi_I \text{ a section of } \mathcal{A}(H_I)\langle\widehat{H_I}\rangle \text{ of degree } p - 2|I|\quad,$$

and its differential $d\varphi$ has components $(d\varphi)_I$ that are

$$(d\varphi)_I = d(\varphi_I) + \sum_{I' \lhd I} R_{I'I}(\varphi_{I'})\,,$$

where we recall $I' \lhd I$ means $I' \subset I$ with $|I'| = |I| - 1$.

The complex $\mathcal{A}(X)_H$ naturally contains the subcomplex $\mathcal{A}(H_I)_{\widehat{H_I}}[-2a]$ for each I of order a. More generally for $J \supset I$, there is an inclusion

$$\mathrm{in}_{I,J} : \mathcal{A}(H_J)_{\widehat{H_J}}[-2|J|] \to \mathcal{A}(H_I)_{\widehat{H_I}}[-2|I|]$$

of complexes on H_I.

The inclusion $e_H : \mathcal{A}(X) \to \mathcal{A}(X)_H$ is a quasi-isomorphism. There is a canonical map $q : \mathcal{A}(X)_H \to \mathcal{A}(X)\langle H\rangle$ obtained by projection; the composition $\mathcal{A}(X) \to \mathcal{A}(X)_H \to \mathcal{A}(X)\langle H\rangle$ coincides with the natural inclusion.

The object $\mathcal{A}(X)_H$ is contravariantly functorial in X. If Z is a smooth closed subvariety meeting H transversally, one has the restriction map $i_Z^* : \mathcal{A}(X) \to \mathcal{A}(Z)$; it extends to a map $i_Z^* : \mathcal{A}(X)\langle H\rangle \to \mathcal{A}(Z)\langle Z \cap H\rangle$.

There are also the restriction maps $i_Z^* : \mathcal{A}(H_I)\langle\widehat{H_I}\rangle \to \mathcal{A}(Z \cap H_I)\langle Z \cap \widehat{H_I}\rangle$, and they commute with the residue maps. Hence the maps

$$i_Z^* : \mathcal{A}(H^{(a)})\langle\widehat{H^{(a)}}\rangle \to \mathcal{A}(Z \cap H^{(a)})\langle Z \cap \widehat{H^{(a)}}\rangle$$

commute with r, giving a map of complexes

$$i_Z^* : \mathcal{A}(X)_H \to \mathcal{A}(Z)_{Z\cap H} .$$

Also the maps $\mathrm{in}_{I,J}$ and i_Z^* commute with each other, namely the diagram (shifts omitted)

$$\begin{array}{ccc} \mathcal{A}(H_J)_{\widehat{H_J}} & \xrightarrow{\mathrm{in}_{I,J}} & \mathcal{A}(H_I)_{\widehat{H_I}} \\ {\scriptstyle i_Z^*}\big\downarrow & & \big\downarrow{\scriptstyle i_Z^*} \\ \mathcal{A}(Z\cap H_J)_{\widehat{Z\cap H_J}} & \xrightarrow{\mathrm{in}_{I,J}} & \mathcal{A}(Z\cap H_I)_{\widehat{Z\cap H_I}} \end{array} \tag{2.3.1}$$

commutes.

Further, the object $\mathcal{A}(X)_H$ behaves in a natural way with respect to enlarging H, see a later subsection for this variance in H.

(2.4) *The complex $\mathcal{A}(X\backslash H)$.* For each b let

$$i_{H*} : \bigoplus_{|J|=b+1} \mathcal{A}(H_J)_{\widehat{H_J}}[-2(b+1)] \to \bigoplus_{|I|=b} \mathcal{A}(H_I)_{\widehat{H_I}}[-2b] \tag{2.4.1}$$

be defined as the sum of $\epsilon(I,J)\cdot \mathrm{in}_{I,J}$; the sign $\epsilon(I,J)$ was defined in (2.2). Then we have $i_{H*}\circ i_{H*} = 0$, and we have a complex of complexes.

Since we have $\mathcal{A}(H^{(b)})_{\widehat{H^{(b)}}} = \bigoplus_{|I|=b} \mathcal{A}(H_I)_{\widehat{H_I}}$, the complex in question is expressed as

$$\cdots \to \mathcal{A}(H^{(b)})_{\widehat{H^{(b)}}}[-2b] \xrightarrow{i_{H*}} \cdots \longrightarrow \mathcal{A}(H^{(1)})_{\widehat{H^{(1)}}}[-2] \xrightarrow{i_{H*}} \mathcal{A}(X)_H \to 0 \tag{2.4.2}$$

(with $\mathcal{A}(X)_H$ in degree 0); recall $H^{(0)} = X$ and $\widehat{H^{(0)}} = H$ by convention, so that $\mathcal{A}(X)_H = \mathcal{A}(H^{(0)})_{\widehat{H^{(0)}}}$. We write $\mathcal{A}(X\backslash H)$ for this complex and for its total complex. It depends on the pair (X,H), not just on the open set $X-H$, and $\mathcal{A}(X\backslash H)$ is just a notation chosen for simplicity. According to (1.3) the differential of $\mathcal{A}(X\backslash H)$ is of the form

$$(-1)^b d_{\mathcal{A}(H^b)_{\widehat{H^b}}} + i_{H*} \quad \text{on} \quad \mathcal{A}(H^b)_{\widehat{H^b}} .$$

A section ξ of $\mathcal{A}(X\backslash H)$ of degree p is a sum $\sum_{b\geq 0}\xi_b$, where ξ_b is a section of $\mathcal{A}(H^b)_{\widehat{H^b}}$ of degree $p-b$. Each ξ_b is a sum $\sum \xi_I$, where I varies over subsets with order b, and $\xi_I \in \mathcal{A}(H_I)_{\widehat{H_I}}$ is an element of degree $p-|I|$.

Each ξ_I in turn is of the form

$$\xi_I = (\xi_{IJ}) ,$$

where J varies over subsets $J\supset I$, and ξ_{IJ} is a section of $\mathcal{A}(H_J)_{\widehat{H_J}}$ of degree $p+|I|-2|J|$. Combining all, one has $\xi = (\xi_{IJ})$, and we call ξ_{IJ} be the IJ-component of ξ.

Using this expression the differential $d\xi$ has the IJ-components given by the formula

$$(d\xi)_{IJ} = (-1)^I \left(d(\xi_{IJ}) + \sum_{J'} R_{J'J}(\xi_{IJ'}) \right) + \sum_{I'} \epsilon(I,I')\xi_{I'J} \quad \text{in} \quad \mathcal{A}(H_J)_{\widehat{H_J}} ,$$

where the index J' in the first sum varies over subsets with $I \subset J' \lhd J$, and I' in the second sum over subsets with $I \lhd I' \subset J$.

One has a canonical injection $\mathcal{A}(X)_H \to \mathcal{A}(X \backslash H)$. Since the composition of i_{H*} with the map $q : \mathcal{A}(X)_H \to \mathcal{A}(X)\langle H\rangle$ is zero, the latter extends to a map of complexes (also written q) $\mathcal{A}(X \backslash H) \to \mathcal{A}(X)\langle H\rangle$.

(2.5) *The map s.* We show that the map $\mathcal{A}(X)_H \to \mathcal{A}(X \backslash H)$ "replaces" the natural map $\mathcal{A}(X) \to \mathcal{A}(X)\langle H\rangle$.

There is a map of complexes

$$s : \mathcal{A}(X)\langle H\rangle \to \mathcal{A}(X \backslash H)\,,$$

which sends a section $\omega \in \mathcal{A}(X)\langle H\rangle$ to the element $s(\omega)$ with IJ-components specified as follows:

$$s(\omega)_{IJ} = \begin{cases} (-1)^{|I|} R_I(\omega) & \text{if } I = J\,, \\ 0 & \text{if } I \neq J\,. \end{cases}$$

The verification that s commutes with d is left to the reader. The map s is a section to $q : \mathcal{A}(X \backslash H) \to \mathcal{A}(X)\langle H\rangle$.

The complexes and the maps we have introduced appear in the following commutative diagram:

$$\begin{tikzcd} \mathcal{A}(X) \arrow[r, hook] \arrow[d, "e_H"'] & \mathcal{A}(X)\langle H\rangle \arrow[d, "s"] \\ \mathcal{A}(X)_H \arrow[r, hook] & \mathcal{A}(X \backslash H) \end{tikzcd} \tag{2.5.1}$$

We also have the next proposition.

(2.6) **Proposition.** *The maps $q : \mathcal{A}(X \backslash H) \to \mathcal{A}(X)\langle H\rangle$ and $s : \mathcal{A}(X)\langle H\rangle \to \mathcal{A}(X \backslash H)$ satisfy $qs = id$ and $sq \simeq id$ (homotopy equivalence); in particular they are quasi-isomorphisms.*

(2.7) *Contravariant functoriality.* Furthering the case for $\mathcal{A}(X)_H$, one shows that the contents of (2.4) and (2.5) are contravariantly functorial in X.

Suppose Z is a smooth closed subvariety meeting H transversally. From the commutativity (2.3.1) we deduce that the maps i_{H*} and i_Z^* commute with each other, namely the diagram

$$\begin{tikzcd} \mathcal{A}(H^{(a+1)})_{H^{(a+1)}}[-2] \arrow[r, "i_{H*}"] \arrow[d, "i_Z^*"'] & \mathcal{A}(H^{(a)})_{H^{(a)}} \arrow[d, "i_Z^*"] \\ \mathcal{A}(Z \cap H^{(a+1)})_{Z \cap H^{(a+1)}}[-2] \arrow[r, "i_{H*}"] & \mathcal{A}(Z \cap H^{(a)})_{Z \cap H^{(a)}} \end{tikzcd} \tag{2.7.1}$$

commutes, so there is an induced map of complexes

$$i_Z^* : \mathcal{A}(X \backslash H) \to \mathcal{A}(Z \backslash Z \cap H)\,. \tag{2.7.2}$$

Let us say that any one of the four maps in the diagram (2.5.1) is a *map of comparison*. Then each map of comparison commutes with the restriction maps i_Z^*. Equivalently one can say that the diagram (2.5.1) be contravariantly functorial. Equivalently still, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{A}(X)\langle H\rangle \arrow[dr] \arrow[dd, "s"'] & \\
& \mathcal{A}(Z) \arrow[rr] \arrow[dd] & & \mathcal{A}(Z)\langle Z\cap H\rangle \arrow[dd, "s"'] \\
\mathcal{A}(X)_H \arrow[rr] \arrow[dr] & & \mathcal{A}(X\backslash H) \arrow[dr] & \\
& \mathcal{A}(Z)_H \arrow[rr] & & \mathcal{A}(Z\backslash Z\cap H)\,.
\end{tikzcd} \tag{2.7.3}$$

(2.8) *Variance in H.* The constructions in subsections (2.3) – (2.5) behave with respect to H as follows.

Suppose H' is a normal crossing divisor with $H \leq H'$; note that $Ind(H) \subset Ind(H')$. Then for $I \in Ind(H)$, one has $H_I = H'_I$ and $\widehat{H_I} \leq \widehat{H'_I}$, hence there is inclusion $\mathcal{A}(H_I)\langle\widehat{H_I}\rangle \subset \mathcal{A}(H'_I)\langle\widehat{H'_I}\rangle$. Also if $I \subset J$ are in $Ind(H)$, then the diagram (shifts omitted) with vertical maps inclusions

$$\begin{array}{ccc}
\mathcal{A}(H_I)\langle\widehat{H_I}\rangle & \xrightarrow{R_{IJ}} & \mathcal{A}(H_J)\langle\widehat{H_J}\rangle \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}(H'_I)\langle\widehat{H'_I}\rangle & \xrightarrow{R_{IJ}} & \mathcal{A}(H'_J)\langle\widehat{H'_J}\rangle
\end{array}$$

commutes. So there is a natural inclusion $e_{H,H'} : \mathcal{A}(X)_H \to \mathcal{A}(X)_{H'}$. If H is the zero divisor, this is the canonical inclusion $e_{H'}$. From the identity $e_{H,H'}e_H = e_{H'}$, it follows that $e_{H,H'}$ is a quasi-isomorphism.

If H'' is another normal crossing divisor with $H' \leq H''$, we have transitivity

$$e_{H,H''} = e_{H',H''}\, e_{H,H'}\,.$$

For subsets $I \subset J$ of $Ind(H)$, the diagram (the vertical maps are the inclusions)

$$\begin{array}{ccc}
\mathcal{A}(H_I)_{\widehat{H_I}} & \xrightarrow{\mathrm{in}_{IJ}} & \mathcal{A}(H_J)_{\widehat{H_J}} \\
\big\downarrow & & \big\downarrow \\
\mathcal{A}(H'_I)_{\widehat{H'_I}} & \xrightarrow{\mathrm{in}_{IJ}} & \mathcal{A}(H'_J)_{\widehat{H'_J}}
\end{array}$$

commutes. Thus there is an inclusion $e_{H,H'} : \mathcal{A}(X\backslash H) \to \mathcal{A}(X\backslash H')$. This also has transitivity for $H \leq H' \leq H''$.

Each map of comparison in (2.5.1) commutes with the map of variance. For the inclusion $\mathcal{A}(X)_H \to \mathcal{A}(X\backslash H)$ this means the commutativity of the diagram

$$\begin{array}{ccc}
\mathcal{A}(X)_H & \longrightarrow & \mathcal{A}(X\backslash H) \\
{\scriptstyle e_{H,H'}}\big\downarrow & & \big\downarrow{\scriptstyle e_{H,H'}} \\
\mathcal{A}(X)_{H'} & \longrightarrow & \mathcal{A}(X\backslash H')\,.
\end{array}$$

Similarly for the map s. Hence, with regard to the commutative diagrams (2.5.1) for H and for H', there is a map of diagrams from the former to the latter.

The maps of variance and contravariant functoriality commute. Specifically the diagram

$$\begin{array}{ccc} \mathcal{A}(X)_H & \xrightarrow{e_{H,H'}} & \mathcal{A}(X)_{H'} \\ {\scriptstyle i_Z^*}\downarrow & & \downarrow{\scriptstyle i_Z^*} \\ \mathcal{A}(Z)_{Z\cap H} & \xrightarrow{e_{H,H'}} & \mathcal{A}(Z)_{Z\cap H'} \end{array}$$

commutes. Similarly for $e_{H,H'} : \mathcal{A}(X\backslash H) \to \mathcal{A}(X\backslash H')$.

(2.9) *The complex* $\mathcal{A}(X|Y)$. Suppose Y is a normal crossing divisor. For $I \subset J$ subsets of $Ind(Y)$, there is the restriction

$$\mathrm{rest}_{I,J} : \mathcal{A}(Y_I) \to \mathcal{A}(Y_J)\,.$$

For subsets I, J with $J \rhd I$ with $I = i_1 \cdots i_a$, $J = i_1 \cdots i_k, j, i_{k+1}, \cdots, i_a$, let $\epsilon(I,J) = (-1)^k$ as before. Define the map

$$i_Y^* : \mathcal{A}(Y^{(b)}) \to \mathcal{A}(Y^{(b+1)}) \tag{2.9.1}$$

to be the sum of $\epsilon(I,J) \cdot \mathrm{rest}_{I,J}$ for $|I| = b$. Consider the complex of complexes

$$0 \to \mathcal{A}(X) \xrightarrow{i^*} \mathcal{A}(Y^{(1)}) \xrightarrow{i^*} \cdots \longrightarrow \mathcal{A}(Y^{(b)}) \longrightarrow \cdots\,, \tag{2.9.2}$$

(with $\mathcal{A}(X)$ in degree 0); the total complex of which we denote by $\mathcal{A}(X|Y)$. The differential of the total complex is

$$(-1)^b d + i^* \quad \text{on} \quad \mathcal{A}(Y^{(b)})\,.$$

Thus a section ψ of $\mathcal{A}(X|Y)$ of degree p is a collection (ψ_I), with I subsets of $Ind(Y)$, where ψ_I is a section of $\mathcal{A}(Y_I)$ of degree $p - |I|$. With this expression $d\psi$ consists of components

$$(d\psi)_I = (-1)^{|I|} d(\psi_I) + \sum_{I' \lhd I} \epsilon(I', I) \cdot \psi_{I'} \quad \text{in} \quad \mathcal{A}(H_I)\,. \tag{2.9.3}$$

With the convention $Y^{(0)} = X$ one may write $\mathcal{A}(Y^{(\bullet)})$ for $\mathcal{A}(X|Y)$. There is a canonical map $\mathcal{A}(X|Y) \to \mathcal{A}(X)$.

Denoting the restriction of $\mathcal{A}_X$ to Y by $\mathcal{A}_X|Y$, one has a complex of complexes

$$0 \to \mathcal{A}_X|Y \to \mathcal{A}(Y^{(1)}) \to \cdots \to \mathcal{A}(Y^{(r)}) \to 0\,,$$

which one verifies to be exact (cf. [Br], II, §13). So the induced map

$$\mathcal{A}_X|Y \to [\mathcal{A}(Y^{(1)}) \to \cdots \to \mathcal{A}(Y^{(r)})]$$

is a quasi-isomorphism of complexes on Y, and the induced map

$$\mathrm{Cone}(\mathcal{A}_X \to \mathcal{A}_X|Y\,)[-1] \to \mathcal{A}(X|Y)$$

is also a quasi-isomorphism.

(2.10) One may combine subsections (2.3) and (2.9) and introduce further variants of the complexes.

1. Suppose given Y a normal crossing divisor meeting H transversally. For each pair of subsets J, J' with $J \rhd J'$, one has restriction $\mathcal{A}(Y_J)_{Y_J \cap H} \to \mathcal{A}(Y_{J'})_{Y_{J'} \cap H}$, see (2.7.2). Taking the sum of them with signs as in (2.9) one obtains the maps

$$i_Y^* : \mathcal{A}(Y^{(b)})_{Y^{(b)} \cap H} \to \mathcal{A}(Y^{(b+1)})_{Y^{(b+1)} \cap H} \,, \tag{2.10.1}$$

which yield a complex of complexes

$$0 \to \mathcal{A}(X)_H \xrightarrow{i_Y^*} \mathcal{A}(Y^{(1)})_{Y^{(1)} \cap H} \xrightarrow{i_Y^*} \cdots \to \mathcal{A}(Y^{(b)})_{Y^{(b)} \cap H} \to \cdots .$$

The total complex of this is denoted by $\mathcal{A}(X|Y)_H$.

There is a canonical map $\mathcal{A}(X|Y)_H \to \mathcal{A}(X)_H$ and a quasi-isomorphism $\mathcal{A}(X|Y) \to \mathcal{A}(X|Y)_H$.

2. With the same notation we have the restriction map $\mathcal{A}(Y_J)\langle Y_J \cap H\rangle \to \mathcal{A}(Y_{J'})\langle Y_{J' \cap H}\rangle$. Taking as i_Y^* the signed sum of them we have a complex of complexes

$$0 \to \mathcal{A}(X)\langle H\rangle \xrightarrow{i_Y^*} \mathcal{A}(Y^{(1)})\langle H \cap Y^{(1)}\rangle \xrightarrow{i_Y^*} \cdots \to \mathcal{A}(Y^{(b)})\langle H \cap Y^{(b)}\rangle \to \cdots ,$$

and its total complex is written $\mathcal{A}(X|Y)\langle H\rangle$.

There is then a map of complexes $\mathcal{A}(X|Y)_H \to \mathcal{A}(X|Y)\langle H\rangle$ obtained by projection.

(2.11) *The complex* $\mathcal{A}(X \backslash H|Y)$. Suppose Y is a normal crossing divisor meeting H transversally. For pairs $I \lhd I'$ and $J \lhd J'$ one has a commutative diagram (see (2.3.1))

$$\begin{array}{ccc}
\mathcal{A}(H_I \cap Y_J)_{\widehat{H_I} \cap Y_J} & \xrightarrow{\mathrm{rest}_{J,J'}} & \mathcal{A}(H_I \cap Y_{J'})_{\widehat{H_I} \cap Y_{J'}} \\
{\scriptstyle \mathrm{in}_{I,I'}}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{in}_{I,I'}} \\
\mathcal{A}(H_{I'} \cap Y_J)_{\widehat{H_{I'}} \cap Y_J} & \xrightarrow{\mathrm{rest}_{J,J'}} & \mathcal{A}(H_{I'} \cap Y_{J'})_{\widehat{H_{I'}} \cap Y_{J'}} \,.
\end{array}$$

Taking signed sums we have commutative squares

$$\begin{array}{ccc}
\mathcal{A}(Y^{(b)} \cap H^{(a)})_{Y^{(b)} \cap \widehat{H^{(a)}}} & \xrightarrow{i_Y^*} & \mathcal{A}(Y^{(b+1)} \cap H^{(a)})_{Y^{(b+1)} \cap \widehat{H^{(a)}}} \\
{\scriptstyle i_{H*}}\big\uparrow & & \big\uparrow{\scriptstyle i_{H*}} \\
\mathcal{A}(Y^{(b)} \cap H^{(a+1)})_{Y^{(b)} \cap \widehat{H^{(a+1)}}}[-2] & \xrightarrow{i_Y^*} & \mathcal{A}(Y^{(b+1)} \cap H^{(a+1)})_{Y^{(b+1)} \cap \widehat{H^{(a+1)}}} \,.
\end{array}$$

Therefore we obtain a double complex of complexes

$$\begin{array}{ccccc}
\mathcal{A}(X)_H & \xrightarrow{i_Y^*} & \mathcal{A}(Y^{(1)})_{Y^{(1)} \cap H} & \xrightarrow{i_Y^*} & \cdots \\
{\scriptstyle i_{H*}}\big\uparrow & & {\scriptstyle i_{H*}}\big\uparrow & & \\
\mathcal{A}(H^{(1)})_{\widehat{H^{(1)}}}[-2] & \xrightarrow{i_Y^*} & \mathcal{A}(Y^{(1)} \cap H^{(1)})_{Y^{(1)} \cap \widehat{H^{(1)}}}[-2] & \xrightarrow{i_Y^*} & \cdots \\
{\scriptstyle i_{H*}}\big\uparrow & & {\scriptstyle i_{H*}}\big\uparrow & & \\
\vdots & & \vdots & &
\end{array} \tag{2.11.1}$$

which has the term

$$\mathcal{A}(Y^{(b)} \cap H^{(a)})_{Y^{(b)} \cap \widehat{H^{(a)}}}[-2a]$$

in bidegree $(b, -a)$; the maps i_Y^* give differential of degree $(1, 0)$ and the map i_{H*} that of degree $(0, 1)$. We write $\mathcal{A}(X \backslash H|Y)$ for this, as well as for its total complex.

The columns are $\mathcal{A}(Y^{(b)} \backslash Y^{(b)} \cap H)$, and one may view diagram (2.11.1) as the total complex of the complex of complexes

$$0 \to \mathcal{A}(X \backslash H) \xrightarrow{i^*} \mathcal{A}(Y^{(1)} \backslash Y^{(1)} \cap H) \xrightarrow{i^*} \cdots$$

There is a canonical surjection $\mathcal{A}(X \backslash H|Y) \to \mathcal{A}(X \backslash H)$.

Likewise the rows are $\mathcal{A}(H^{(a)}|H^{(a)} \cap Y)_{\widehat{H^{(a)}}}[-2a]$, and (2.11.1) may be viewed as

$$\cdots \xrightarrow{i_*} \mathcal{A}(H^{(1)}|H^{(1)} \cap Y)_{\widehat{H^{(1)}}}[-2] \xrightarrow{i_*} \mathcal{A}(X|Y)_H \to 0\,.$$

There is a canonical injection $\mathcal{A}(X|Y)_H \to \mathcal{A}(X \backslash H|Y)$.

Generalizing (2.5.1) there is a commutative diagram

$$\begin{tikzcd}
\mathcal{A}(X|Y) \arrow[r, hook] \arrow[d] & \mathcal{A}(X|Y)\langle H\rangle \arrow[d, "s"] \\
\mathcal{A}(X|Y)_H \arrow[r, hook] & \mathcal{A}(X \backslash H|Y)\,.
\end{tikzcd} \tag{2.11.2}$$

The four canonical maps such as $\mathcal{A}(X|Y) \to \mathcal{A}(X)$ from the vertices of (2.5.1) to the corresponding vertices of (2.10.2) form a commutative diagram:

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[rr] \arrow[dd] & & \mathcal{A}(X)\langle H\rangle \arrow[dd, "s"'] & \\
& \mathcal{A}(X|Y) \arrow[ul] \arrow[rr] \arrow[dd] & & \mathcal{A}(X|Y)\langle H\rangle \arrow[ul] \arrow[dd, "s"] \\
\mathcal{A}(X)_H \arrow[rr] & & \mathcal{A}(X \backslash H) & \\
& \mathcal{A}(X|Y)_H \arrow[ul] \arrow[rr] & & \mathcal{A}(X \backslash H|Y)\,. \arrow[ul]
\end{tikzcd} \tag{2.11.3}$$

Note that all the variants of $\mathcal{A}(X)$ introduced thus far appear in this diagram. The verification of commutativity is straightforward. One also has the following proposition.

(2.12) **Proposition.** *The maps $q : \mathcal{A}(X \backslash H|Y) \to \mathcal{A}(X|Y)\langle H\rangle$ and $s : \mathcal{A}(X|Y)\langle H\rangle \to \mathcal{A}(X \backslash H|Y)$ satisfy $qs = id$ and $sq \simeq id$ (homotopy equivalence); in particular they are quasi-isomorphisms.*

(2.13) The objects and contents of (2.9) – (2.11) are contravariantly functorial in X, and have expected variance in H and in Y.

Suppose Z is a smooth closed subvariety transversal to Y; there is then a map $i_Z^* : \mathcal{A}(X|Y) \to \mathcal{A}(Z|Z \cap Y)$. If Z meets $Y + H$ transversally, then there are maps of complexes $i_Z^* : \mathcal{A}(X|Y)\langle H\rangle \to \mathcal{A}(Z|Z \cap Y)\langle Z \cap H\rangle$,

$$i_Z^* : \mathcal{A}(X|Y)_H \to \mathcal{A}(Z|Z \cap Y)_{Z \cap H}\,.$$

and

$$i_Z^* : \mathcal{A}(X\backslash H|Y) \to \mathcal{A}(Z\backslash Z\cap H|Z\cap Y)\,.$$

Each comparison map in (2.11.2) is compatible with i_Z^*, and the commutative diagram (2.7.3) generalizes accordingly.

The complex $\mathcal{A}(X|Y)_H$ has variance for $H \le H'$, and the same holds for the complex $\mathcal{A}(X\backslash H|Y)$. The details are omitted.

Suppose now Y' is another normal crossing divisor with $Y \le Y'$. Then we have a natural map of complexes $e_{Y,Y'} : \mathcal{A}(X|Y') \to \mathcal{A}(X|Y)$; it is transitive for $Y \le Y' \le Y''$. One also has similar variance in Y for the complexes $\mathcal{A}(X|Y)_H$ and $\mathcal{A}(X\backslash H|Y)$.

(2.14) There are further variants of the complexes. Suppose H' is another normal crossing divisor meeting H transversally. Then one has the complex $\mathcal{A}(X\backslash H)_{H'}$, a variant of $\mathcal{A}(X\backslash H)$, which is of the form

$$\cdots \to \mathcal{A}(H^{(b)})_{\widehat{H^{(b)}}+H^{(b)}\cap H'}[-2b] \xrightarrow{i_{H*}} \cdots \longrightarrow \mathcal{A}(H^{(1)})_{\widehat{H^{(1)}}+H^{(1)}\cap H'}[-2] \xrightarrow{i_{H*}} \mathcal{A}(X)_{H+H'} \to 0$$

Likewise if $Y+H$ is transversal to H' one has the complex $\mathcal{A}(X\backslash H|Y)_{H'}$ which is a variant of $\mathcal{A}(X\backslash H|Y)$.

(2.15) With assumptions as in (2.3), let $U = X - H$. The complexes $\mathcal{A}(X)$, $\mathcal{A}(X)_H$, $\mathcal{A}(X)\langle H\rangle$ and $\mathcal{A}(X\backslash H)$ appearing in diagrams (2.5.1) all restrict to $\mathcal{A}_U$ on U, and all the maps appearing in (2.5.1) restrict to the identity. Thus the induced maps from the four complexes to $j_*\mathcal{A}_U$ all commute with the maps, resulting in commutative diagrams

$$\begin{tikzcd} \mathcal{A}(X) \arrow[r, hook] \arrow[d, hook, "in"'] & \mathcal{A}(X)\langle H\rangle \arrow[r] \arrow[d, "s"] & j_*\mathcal{A}_U \arrow[d, "id"] \\ \mathcal{A}(X)_H \arrow[r, hook] & \mathcal{A}(X\backslash H) \arrow[r] & j_*\mathcal{A}_U\,. \end{tikzcd} \tag{2.15.1}$$

More generally, in the presence of a normal crossing divisor Y that meet H transversally, introduce the complex

$$\mathcal{A}_{U|Y\cap U} := \mathrm{Tot}\,[\mathcal{A}_U \to \mathcal{A}_{U\cap Y^{(1)}} \to \cdots]\,.$$

Then there are induced maps $\mathcal{A}(X|Y)\langle H\rangle \to j_*\mathcal{A}_{U|Y\cap U}$ and $\mathcal{A}(X\backslash H|Y) \to j_*\mathcal{A}_{U|Y\cap U}$, that extend commutative diagrams (2.10.2). For example the left diagram extends to

$$\begin{tikzcd} \mathcal{A}(X|Y) \arrow[r, hook] \arrow[d, hook] & \mathcal{A}(X|Y)\langle H\rangle \arrow[r] \arrow[d, "s"] & j_*(\mathcal{A}_{U|Y\cap U}) \arrow[d, "id"] \\ \mathcal{A}(X|Y)_H \arrow[r, hook] & \mathcal{A}(X\backslash H|Y) \arrow[r] & j_*(\mathcal{A}_{U|Y\cap U})\,. \end{tikzcd} \tag{2.15.2}$$

Complexes of sheaves of currents.

We recall the notion of dual cosheaf from [Br], Chap.V, §1. For a c-soft sheaf $\mathcal{L}$, its dual cosheaf, denoted by $\Gamma_c\{\mathcal{L}\}$, is given by

$$V \mapsto \Gamma_c(V, \mathcal{L}) .$$

When $\mathcal{L}$ is a sheaf of $\mathbb{C}$-vector spaces, this implies that $V \mapsto \Gamma_c(V, \mathcal{L})^*$ is a sheaf.

If $\mathcal{L}^\bullet$ is a complex of c-soft sheaves, then $\Gamma_c\{\mathcal{L}^\bullet\}$ is a complex of cosheaves, and $V \mapsto \Gamma_c(V, \mathcal{L}^\bullet)^*$ is a complex of sheaves; we denote this by $D(\mathcal{L}^\bullet)$. The functor $\mathcal{L}^\bullet \mapsto \Gamma_c\{\mathcal{L}^\bullet\}$ is exact and takes quasi-isomorphisms to quasi-isomorphisms.

(2.16) Let $\mathcal{D}(X) = D(\mathcal{A}(X))[-2n]$, the dual of the complex $\mathcal{A}(X)$ on X.

If Z is a smooth closed subvariety of codimension c, the restriction map $i_Z^* : \mathcal{A}(X) \to \mathcal{A}(Z)$ induces a map $i_{Z*} : \mathcal{D}(Z)[-2c] \to \mathcal{D}(X)$.

We will successively introduce variants of $\mathcal{D}(X)$, taking duals of variants of $\mathcal{A}(X)$ in the previous subsections. Along the way the facts we have obtained for the variants of $\mathcal{A}(X)$ will also be dualized.

1. Let $\mathcal{D}(X)_H = D(\mathcal{A}(X)_H)[-2n]$, which is of the form

$$\cdots \xrightarrow{r'} D(\mathcal{A}(H^{(2)})\langle\widehat{H^{(2)}}\rangle[-2]\,) \xrightarrow{r'} D(\mathcal{A}(H^{(1)})\langle\widehat{H^{(1)}}\rangle[-1]\,) \xrightarrow{r'} D(\mathcal{A}(X)\langle H\rangle) \to 0$$

where r' are the duals of the maps r in (2.3), and the term $D(\mathcal{A}(X)\langle H\rangle)$ is in degree $2n$.

The quasi-isomorphism $e_H : \mathcal{A}(X) \to \mathcal{A}(X)_H$ induces a quasi-isomorphism

$$e'_H : \mathcal{D}(X)_H \to \mathcal{D}(X) .$$

2. The map $\mathrm{in}_{I,J} : \mathcal{A}(H_J)_{\widehat{H_J}}[-2|J|] \to \mathcal{A}(H_I)_{\widehat{H_I}}[-2|I|]$ in (2.3) induces a map $\mathrm{in}'_{I,J} : \mathcal{D}(H_I)_{\widehat{H_I}} \to \mathcal{D}(H_J)_{\widehat{H_J}}$. Hence the maps $i_{H*} : \mathcal{A}(H^{(a+1)})_{\widehat{H^{(a+1)}}}[-2] \to \mathcal{A}(H^{(a)})_{\widehat{H^{(a)}}}$ induce the maps

$$i_H^* : \mathcal{D}(H^{(a)})_{\widehat{H^{(a)}}} \to \mathcal{D}(H^{(a+1)})_{\widehat{H^{(a+1)}}} .$$

We define $\mathcal{D}(X|H) = D(\mathcal{A}(X\backslash H))[-2n]$ which is of the form

$$0 \to \mathcal{D}(X)_H \xrightarrow{i_H^*} \mathcal{D}(H^{(1)})_{H^{(1)}} \xrightarrow{i_H^*} \cdots .$$

There is a canonical surjection $\mathcal{D}(X|H) \to \mathcal{D}(X)_H$.

The map $q : \mathcal{A}(X\backslash H) \to \mathcal{A}(X)\langle H\rangle$ in (2.5) induces a map $q' : D(\mathcal{A}(X)\langle H\rangle)[-2n] \to \mathcal{D}(X|H)$. The map s induces $s' : \mathcal{D}(X|H) \to D(\mathcal{A}(X)\langle H\rangle)[-2n]$.

There is a commutative diagrams dual to (2.5.1):

$$\begin{tikzcd} \mathcal{D}(X) & D(\mathcal{A}(X)\langle H\rangle)[-2n] \arrow[l] \\ \mathcal{D}(X)_H \arrow[u, "e'_H"] & \mathcal{D}(X|H) \arrow[l] \arrow[u, "s'"'] \end{tikzcd} \tag{2.16.1}$$

Proposition (2.6) has its dual statement. In particular s' is a quasi-isomorphism.

Dualize (2.7): Under the same assumption, the diagram (2.7.1) may be dualized. One has maps

$$i_{Z*} : \mathcal{D}(Z)_{Z\cap H}[-2c] \to \mathcal{D}(X)_H$$

and

$$i_{Z*} : \mathcal{D}(Z|Z\cap H)[-2c] \to \mathcal{D}(X|H)\,.$$

The diagram (2.16.1) is covariantly functorial, in other words there is a commutative diagram dual to (2.7.3).

Dualize (2.8): Under the same condition, one has a map of variance $e_{H,H'} : \mathcal{D}(X)_{H'} \to \mathcal{D}(X)_H$, a quasi-isomorphism, and a map

$$e_{H,H'} : \mathcal{D}(X|H') \to \mathcal{D}(X|H)\,.$$

Both maps are transitive in H.

Each map of comparison in the diagram (2.16.1) commutes with a map of variance. A map of variance and covariant functoriality commutes.

3. The maps $i_Y^* : \mathcal{A}(Y^{(b)}) \to \mathcal{A}(Y^{(b+1)})$ in (2.4) induce maps $i_{Y*} : \mathcal{D}(Y^{(b+1)}) \to \mathcal{D}(Y^{(b)})$. We define $\mathcal{D}(X\backslash Y) = D(\mathcal{A}(X|Y))[-2n]$, which is of the form

$$\cdots \xrightarrow{i_*} \mathcal{D}(Y^{(1)})[-2] \xrightarrow{i_*} \mathcal{D}(X) \to 0\,.$$

4. The map $i_Y^* : \mathcal{A}(Y^{(b)})_H \to \mathcal{A}(Y^{(b+1)})_H$ in (2.10) induce maps $i_{Y*} : \mathcal{D}(Y^{(b+1)})_H \to \mathcal{D}(Y^{(b)})_H$. Let $\mathcal{D}(X\backslash Y)_H = D(\mathcal{A}(X|Y)_H)[-2n]$, which is of the form

$$\cdots \xrightarrow{i_*} \mathcal{D}(Y^{(1)})_{H\cap Y^{(1)}}[-2] \xrightarrow{i_*} \mathcal{D}(X)_H \to 0\,.$$

5. The dual of the double complex $\mathcal{A}(X\backslash H|Y)$ looks like, after the shift $[-2n]$,

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \uparrow{\scriptstyle i_H^*} & & \uparrow{\scriptstyle i_H^*} \\
\cdots & \xrightarrow{i_{Y*}} & \mathcal{D}(Y^{(1)}\cap H^{(1)})_{Y^{(1)}\cap \widehat{Y^{(1)}}}[-2] & \xrightarrow{i_{Y*}} & \mathcal{D}(H^{(1)})_{\widehat{H^{(1)}}} \\
 & & \uparrow{\scriptstyle i_H^*} & & \uparrow{\scriptstyle i_H^*} \\
\cdots & \xrightarrow{i_{Y*}} & \mathcal{D}(Y^{(1)})_{Y^{(1)}\cap H}[-2] & \xrightarrow{i_{Y*}} & \mathcal{D}(X)_H\,.
\end{array}$$

It has terms

$$\mathcal{D}(Y^{(b)}\cap H^{(a)})_{Y^{(b)}\cap \widehat{H^{(a)}}}[-2a]$$

in degree $(-b,a)$, $a,b\geq 0$, and the two differentials sums of i_* and i^*. This we denote this by $\mathcal{D}(X\backslash Y|H)$, namely

$$\mathcal{D}(X\backslash Y|H) = D(\mathcal{A}(X\backslash H|Y))[-2n]\,.$$

It has 0-th row equal to $\mathcal{D}(X\backslash Y)_H$ and 0-th column equal to $\mathcal{D}(X|H)$.

Dualizing (2.11.2) we obtain a commutative a diagram

$$\begin{tikzcd}
\mathcal{D}(X\backslash Y) & D(\mathcal{A}(X|Y)\langle H\rangle)[-2n] \arrow[l] \\
\mathcal{D}(X\backslash Y)_H \arrow[u] & \mathcal{D}(X\backslash Y|H)\,. \arrow[l] \arrow[u, "s'"']
\end{tikzcd} \tag{2.16.2}$$

There is also a commutative diagram dual to (2.11.3).

Just as contravariant functoriality $i_Z^*: \mathcal{A}(X) \to \mathcal{A}(Z)$ induces covariant functoriality for the complexes $\mathcal{D}(X)$, other instances of the maps i_Z^* for variants of the complexes of $\mathcal{A}(X)$ induce maps i_{Z*} for the corresponding dual complexes. They are compiled in the next table where maps on the left induce maps on the right.

$i_Z^*: \mathcal{A}(X)_H \to \mathcal{A}(Z)_{Z\cap H}$	$i_{Z*}: \mathcal{D}(Z)_{Z\cap H}[-2c] \to \mathcal{D}(X)_H$
$i_Z^*: \mathcal{A}(X\backslash H) \to \mathcal{A}(Z\backslash Z\cap H)$	$i_{Z*}: \mathcal{D}(Z|Z\cap H)[-2c] \to \mathcal{D}(X|Y)$
$i_Z^*: \mathcal{A}(X|Y) \to \mathcal{A}(Z|Z\cap Y)$	$i_{Z*}: \mathcal{D}(Z\backslash Z\cap H)[-2c] \to \mathcal{D}(X\backslash Y)$
$i_Z^*: \mathcal{A}(X|Y)\langle H\rangle \to \mathcal{A}(Z|Z\cap Y)\langle Z\cap H\rangle$	$i_{Z*}: D(\mathcal{A}(Z|Z\cap Y)\langle Z\cap H\rangle) \to D(\mathcal{A}(X|Y)\langle H\rangle)$
$i_Z^*: \mathcal{A}(X|Y)_H \to \mathcal{A}(Z|Z\cap Y)_{Z\cap H}$	$i_{Z*}: \mathcal{D}(Z\backslash Z\cap H)_{Z\cap H}[-2c] \to \mathcal{D}(X\backslash Y)_Z$
$i_Z^*: \mathcal{A}(X\backslash H|Y) \to \mathcal{A}(Z\backslash Z\cap H|Z\cap Y)$	$i_{Z*}: \mathcal{D}(Z\backslash Z\cap Y|Z\cap H)[-2c] \to \mathcal{D}(X\backslash Y|H)\,.$

The complexes $\mathcal{D}(X\backslash Y)_H$ and $\mathcal{D}(X\backslash Y|H)$ have variance in H, induced from variance of the complexes $\mathcal{A}(X|Y)_H$ and $\mathcal{A}(X\backslash H|Y)$. Similarly for the variance in Y.

3 The maps Φ and $\mathcal{P}$

(3.1) Let

$$\Phi^{(0)}: \mathcal{C}(X)_H \to D(\mathcal{A}(X)\langle H\rangle)[-2n]$$

be the map of degree zero given by

$$\langle \Phi^{(0)}(\alpha), \varphi\rangle = \int_\alpha \varphi\,.$$

This is not a map of complexes; one has the equality

$$\Phi^{(0)}(\partial\alpha) = \delta\Phi^{(0)}(\alpha) + r'\Phi^{(0)}(\alpha|H^{(0)})\,, \tag{3.1.1}$$

where r' is the dual of the map $r: \mathcal{A}(X)\langle H\rangle \to \mathcal{A}(H^{(0)})\langle\widehat{H^{(0)}}\rangle[-1]$.

We have likewise maps $\Phi^{(a)}: \mathcal{C}(H^{(a)})_{\widehat{H^{(a)}}} \to D(\mathcal{A}(H^{(a)}))\langle\widehat{H^{(a)}}\rangle[-2(n-a)]$. Define now a map

$$\Phi: \mathcal{C}(X)_H \to \mathcal{D}(X)_H[-2n]$$

to be the one given by

$$\Phi(\alpha) = \left(\Phi^{(a)}(\alpha|H^{(a)})\right)_{a\geq 0} .$$

This is a map of complexes, as can be verified using (3.1.1).

(3.2) **Proposition.** *The diagram*

$$\begin{array}{ccc} \mathcal{C}(X)_H & \xrightarrow{\Phi} & \mathcal{D}(X)_H \\ {\scriptstyle i_H{}^*}\big\downarrow & & \big\downarrow{\scriptstyle i_H{}^*} \\ \mathcal{C}(H^{(1)})_{\widehat{H^{(1)}}} & \xrightarrow{\Phi} & \mathcal{D}(H^{(1)})_{\widehat{H^{(1)}}} \end{array}$$

commutes.

Let $Z \subset X$ be a smooth closed subvariety of codimension d. Then the diagram

$$\begin{array}{ccc} \mathcal{C}(X)_H & \xrightarrow{\Phi} & \mathcal{D}(X)_H \\ {\scriptstyle i_{Z*}}\big\uparrow & & \big\uparrow{\scriptstyle i_{Z*}} \\ \mathcal{C}(Z)_{Z\cap H}[-2d] & \xrightarrow{\Phi} & \mathcal{D}(Z)_{Z\cap H}[-2d] \end{array}$$

commutes.

(3.3) For each J and I one has a map of complexex

$$\Phi : \mathcal{C}(Y_J \cap H_I)_{Y_J\cap\widehat{H_I}} \to \mathcal{D}(Y_J \cap H_I)_{Y_J\cap\widehat{H_I}}$$

defined by integration, hence there are maps

$$\Phi : \mathcal{C}(Y^{(b)} \cap H^{(a)})_{Y^{(b)}\cap\widehat{H^{(a)}}} \to \mathcal{D}(Y^{(b)} \cap H^{(a)})_{Y^{(b)}\cap\widehat{H^{(a)}}}$$

for (b, a). These are compatible with the maps i_* and i^* in the respective double complexes for $\mathcal{C}(X\backslash Y|H)$ and $\mathcal{D}(X\backslash Y|H)$. They give a map of complexes

$$\Phi : \mathcal{C}(X\backslash Y|H) \to \mathcal{D}(X\backslash Y|H)$$

(3.4) Suppose for simplicity all sheaves are ones of $\mathbb{C}$-vector spaces. If $\mathcal{A}$ is another sheaf, by a *pairing* $\mathcal{A} \otimes \Gamma_c\{\mathcal{L}\} \to \mathbb{C}$ we mean a collection of maps

$$f_V : \Gamma(V, \mathcal{A}) \otimes \Gamma_c(V, \mathcal{L}) \to \mathbb{C}$$

for each V, such that for a smaller open set W, the maps f_V and f_W are commute via restriction in the first variable and corestriction in the second variable, namely that the digram

$$\begin{array}{ccccc} \Gamma(V, \mathcal{A}) & \otimes & \Gamma_c(V, \mathcal{L}) & \to & \mathbb{C} \\ \big\downarrow & & \big\uparrow & & \| \\ \Gamma(W, \mathcal{A}) & \otimes & \Gamma_c(W, \mathcal{L}) & \to & \mathbb{C} \end{array}$$

commutes.

When $\mathcal{A}^\bullet$ is a complex of sheaves, one can speak of a paring $\mathcal{A}^\bullet \otimes \Gamma_c\{\mathcal{L}^\bullet\} \to \mathbb{C}$, which by definition is a collection of parings of complexes $f_V : \Gamma(V, \mathcal{A}^\bullet) \otimes \Gamma_c(V, \mathcal{L}^\bullet) \to \mathbb{C}$ satisfying commutativity as above with respect to restriction and corestriction. A pairing induces a map of complexes of sheaves $\mathcal{A}^\bullet \to D(\mathcal{L}^\bullet)$, see (1.1) for the sign to be attached.

(3.5) *Poincaré duality pairings.* For each open set V of X, one has the canonical pairing

$$\langle\ ,\ \rangle : \Gamma(V, \mathcal{A}_X) \otimes \Gamma_c(V, \mathcal{A}_X)[2n] \to \mathbb{C} \tag{3.5.1}$$

given for ω of degree i and φ of degree $2n-i$ by

$$\langle \omega, \varphi \rangle = \int_V \omega \wedge \varphi\,.$$

The integral makes sense since φ has compact support.

This pairing is compatible with respect to restrictions to smaller open sets, namely if W is a smaller open set, then the above pairing and the paring on W commute via the restriction $\Gamma(V, \mathcal{A}_X) \to \Gamma(W, \mathcal{A}_X)$ and corestriction $\Gamma_c(W, \mathcal{A}_X) \to \Gamma_c(V, \mathcal{A}_X)$. We have thus a pairing $\mathcal{A}_X \otimes \Gamma_c\{\mathcal{A}_X\}[2n] \to \mathbb{C}$.

It induces a map of complexes of sheaves

$$\mathcal{P} : \mathcal{A}(X) \to \mathcal{D}(X)$$

which sends a form ω of degree i on V to the section $(-1)^{s(i)} \cdot [\omega]$ of $\mathcal{D}(X)$ on V defined by $[\omega](\varphi) = \langle \omega, \varphi \rangle$. It is a quasi-isomorphism.

There is also a pairing

$$\langle\ ,\ \rangle : \Gamma_c(V, \mathcal{A}_X) \otimes \Gamma(V, \mathcal{A}_X)[2n] \to \mathbb{C}$$

given by the same formula. This is, however, no different from the paring (3.5.1). Indeed one verifies that this is obtained from (3.5.1) by means of the isomorphism σ that exchanges the factors; note that for ω of degree i and φ of degree $2n-i$, one has $\omega \wedge \varphi = (-1)^i \varphi \wedge \omega$.

(3.6) We introduce variants of the above. Let H and Y be normal crossing divisors on X which meet transversally. One has a pairing (for V open)

$$\langle\ ,\ \rangle : \Gamma(V, \mathcal{A}(X)_H) \otimes \Gamma_c(V, \mathcal{A}(X)_Y)[2n] \to \mathbb{C} \tag{3.6.1}$$

given as follows. Take a section $\psi = (\psi_I)$ of $\mathcal{A}(X)_H$ of degree i, and a section $\varphi = (\varphi_J)$ of $\mathcal{A}(X)_Y$ of degree $2n-i$. We define

$$\langle \psi, \varphi \rangle = \sum_{I,J} \int_{V \cap H_I \cap Y_J} \psi_I \wedge \varphi_J\,.$$

This is a map of complexes. Also this clearly commutes with restriction and corestriction, and defines a paring $\mathcal{A}(X)_H \otimes \Gamma_c\{\mathcal{A}(X)_Y\} \to \mathbb{C}$.

It gives a map of complexes

$$\mathcal{P} : \mathcal{A}(X)_H \to \mathcal{D}(X)_Y\,. \tag{3.6.2}$$

For a section $\psi = (\psi_I) \in \Gamma(V, \mathcal{A}(X)_H)$, the components of $\mathcal{P}(\psi)$ are given by

$$\mathcal{P}(\psi)_J = \sum_I [\psi_I|_{H_I \cap Y_J}]\,;$$

here ψ_I restricts to the form $\psi_I|_{H_I \cap Y_J}$ on $V \cap H_I \cap Y_J$, which determines the current $[\psi|_{H_I \cap Y_J}] \in \Gamma(V \cap H_I \cap Y_J, \mathcal{D}(H_I \cap Y_J)\,)$, which we view as an element in $\Gamma(V \cap Y_J, \mathcal{D}(Y_J)\,)$.

As a particular case where H and Y are empty, we have

$$\mathcal{P} : \mathcal{A}(X) \to \mathcal{D}(X)\,.$$

One verifies that the diagram

$$\begin{array}{ccc} \mathcal{A}(X) & \xrightarrow{\mathcal{P}} & \mathcal{D}(X) \\ \downarrow & & \uparrow \\ \mathcal{A}(X)_H & \xrightarrow{\mathcal{P}} & \mathcal{D}(X)_Y \end{array}$$

commutes.

(3.7) For a normal crossing divisor H, there is a pairing

$$\langle\ ,\ \rangle : \Gamma(V, \mathcal{A}(X)\langle H\rangle) \otimes \Gamma_c(V, \mathcal{A}(X\,|\,H)\,)[2n] \to \mathbb{C} \tag{3.7.1}$$

given by

$$\langle \omega, (\varphi_I)\,\rangle = \sum_I s(\omega, I) \int_{H_I} R_{H_I}(\omega) \wedge \varphi_I\,,$$

with $s(\omega, I) := (-1)^{ia + a(a-1)/2}$, for ω of degree i and (φ_I) of degree $2n - i$. The verification of the condition of pairing is left to the reader.

(3.8) Suppose H and Y are normal crossing divisors meeting transversally. For each I (of order a) one has a pairing

$$\langle\ ,\ \rangle_I : \Gamma(V, \mathcal{A}(H_I)_{H_I \cap Y}) \otimes \Gamma_c(V, \mathcal{A}(H_I)_{\widehat{H_I}}\,)[2(n-a)] \to \mathbb{C}$$

For $J \rhd I$ the pairings for H_I and for H_J commute (shifts are omitted):

$$\begin{array}{ccccc} \Gamma(V, \mathcal{A}(H_I)_{H_I \cap Y}) & \otimes & \Gamma_c(V, \mathcal{A}(H_I)_{\widehat{H_I}}) & \longrightarrow & \mathbb{C} \\ {\scriptstyle i^*}\downarrow & & \uparrow{\scriptstyle i_*} & & \| \\ \Gamma(V, \mathcal{A}(H_J)_{H_J \cap Y}) & \otimes & \Gamma_c(V, \mathcal{A}(H_J)_{\widehat{H_J}}) & \longrightarrow & \mathbb{C} \end{array}$$

For each $a \geq 0$ we have thus a paring

$$\langle\ ,\ \rangle_a : \Gamma(V, \mathcal{A}(H^{(a)})_Y\,) \otimes \Gamma_c(V, \mathcal{A}(H^{(a)})_{\widehat{H^{(a)}}}\,)[2(n-a)] \to \mathbb{C}$$

and the parings for a and for $a+1$ are commute via the maps i_H^* and i_{H*}, since the signs used for the two maps, (2.4) and (2.9), are equal. Therefore we have an induced pairing as in (1.4)

$$\Gamma(V, \mathcal{A}(X|H)_Y) \otimes \Gamma_c(V, \mathcal{A}(X\backslash H))[2n] \to \mathbb{C}\,.$$

Because of commutativity with respect to restriction and corestriction, one has a pairing

$$\mathcal{A}(X|H) \otimes \Gamma_c\{\mathcal{A}(X\backslash H)\}[2n] \to \mathbb{C}\,. \tag{3.8.1}$$

It induces a map of complexes

$$\mathcal{P} : \mathcal{A}(X|H)_Y \to \mathcal{D}(X|H)\,. \tag{3.8.1$'$}$$

An alternative way to obtain this last map is this: there is a map of complexes of complexes

$$\begin{array}{ccccccc}
0 \to & \mathcal{A}(X)_Y & \longrightarrow & \mathcal{A}(H^{(1)})_{H^{(1)}\cap Y} & \longrightarrow & \mathcal{A}(H^{(2)})_{H^{(2)}\cap Y} & \to \cdots \\
& {\scriptstyle\mathcal{P}}\big\downarrow & & {\scriptstyle\mathcal{P}}\big\downarrow & & {\scriptstyle\mathcal{P}}\big\downarrow & \\
0 \to & \mathcal{D}(X)_H & \longrightarrow & \mathcal{D}(H^{(1)})_{\widehat{H^{(1)}}} & \longrightarrow & \mathcal{D}(H^{(2)})_{\widehat{H^{(2)}}} & \to \cdots
\end{array}$$

from which the map results by totalization.

Starting instead with the pairing $\Gamma(V, \mathcal{A}(H_I)_{\widehat{H_I}}) \otimes \Gamma_c(V, \mathcal{A}(H_I)_{H_I\cap Y})[2(n-a)] \to \mathbb{C}$, we obtain another paring

$$\Gamma(V, \mathcal{A}(X\backslash H)) \otimes \Gamma_c(V, \mathcal{A}(X|H)_Y)[2n] \to \mathbb{C}\,.$$

and hence a paring

$$\mathcal{A}(X\backslash H) \otimes \Gamma_c\{\mathcal{A}(X|H)_Y\}[2n] \to \mathbb{C}\,. \tag{3.8.2}$$

and thence a map of complexes

$$\mathcal{P} : \mathcal{A}(X\backslash H) \to \mathcal{D}(X\backslash H)_Y\,. \tag{3.8.2$'$}$$

This map also arises from the following map of complexes of complexes

$$\begin{array}{ccccc}
\cdots & \longrightarrow & \mathcal{A}(H^{(1)})_{\widehat{H^{(1)}}} & \longrightarrow & \mathcal{A}(X)_H \quad \to 0 \\
& & {\scriptstyle\mathcal{P}}\big\downarrow & & {\scriptstyle\mathcal{P}}\big\downarrow \\
\cdots & \longrightarrow & \mathcal{D}(H^{(1)})_{H^{(1)}\cap Y} & \longrightarrow & \mathcal{D}(X)_Y \quad \to 0\,.
\end{array}$$

In the next proposition we compare the latter pairing (3.8.2) for Y the zero divisor with the paring in the previous subsection.

(3.9) **Proposition.** *The pairings defined in the previous subsections are compatible, namely the following diagram commutes:*

$$\begin{array}{ccccc}
\Gamma(V, \mathcal{A}(X)\langle H\rangle) & \otimes & \Gamma_c(V, \mathcal{A}(X\backslash H)[2n] & \longrightarrow & \mathbb{C} \\
{\scriptstyle s}\big\downarrow & & \big\downarrow{\scriptstyle id} & & \| \\
\Gamma(V, \mathcal{A}(X\backslash H))\,) & \otimes & \Gamma_c(V, \mathcal{A}(X|H)[2n] & \longrightarrow & \mathbb{C}\,.
\end{array}$$

(3.10) For a pair of subsets I, J consider the variety $H_I \cap Y_J$. There is a paring

$$\Gamma(V, \mathcal{A}(H_I \cap Y_J)_{\widehat{H_I \cap Y_J}}) \otimes \Gamma_c(V, \mathcal{A}(H_I \cap Y_J)_{H_I \cap \widehat{Y_J}}) \to \mathbb{C}.$$

For each pair of integers a, b, the sum of these give a paring

$$\Gamma(V, \mathcal{A}(H^{(a)} \cap Y^{(b)})_{\widehat{H^{(a)}} \cap Y^{(b)}}) \otimes \Gamma_c(V, \mathcal{A}(H^{(a)} \cap Y^{(b)})_{H^{(a)} \cap \widehat{Y^{(b)}}}) \to \mathbb{C}.$$

It is compatible with the i_{H*} and i_H^*, meaning the commutativity of the diagram of pairings:

$$\begin{array}{ccccc}
\Gamma(V, \mathcal{A}(H^{(a)} \cap Y^{(b)})_{\widehat{H^{(a)}} \cap Y^{(b)}}) & \otimes & \Gamma_c(V, \mathcal{A}(H^{(a)} \cap Y^{(b)})_{H^{(a)} \cap \widehat{Y^{(b)}}}) & \longrightarrow & \mathbb{C} \\
{\scriptstyle i_{H*}}\Big\uparrow & & \Big\downarrow{\scriptstyle i_H^*} & & \| \\
\Gamma(V, \mathcal{A}(H^{(a+1)} \cap Y^{(b)})_{\widehat{H^{(a+1)}} \cap Y^{(b)}}) & \otimes & \Gamma_c(V, \mathcal{A}(H^{(a+1)} \cap Y^{(b)})_{H^{(a+1)} \cap \widehat{Y^{(b)}}}) & \longrightarrow & \mathbb{C}\,.
\end{array}$$

The same holds with respect to the maps i_{Y*} and i_Y^*. We have thus a pairing of double complex of complexes

$$\Gamma(V, \mathcal{A}(X \backslash H | Y)) \otimes \Gamma_c(V, \mathcal{A}(X \backslash Y | H))[2n] \to \mathbb{C}\,,$$

which are compatible with respect to restriction and corestriction. Hence results an induced map of complexes

$$\mathcal{P} : \mathcal{A}(X \backslash H | Y) \to \mathcal{D}(X \backslash H | Y)\,. \tag{3.11.1}$$

This extends the maps $\mathcal{A}(X \backslash H) \to \mathcal{D}(X \backslash H)_Y$ and $\mathcal{A}(X|Y)_H \to \mathcal{D}(X|Y)$ introduced just above.

One verifies that the square

$$\begin{array}{ccc}
\mathcal{A}(X)_H & \xrightarrow{\mathcal{P}} & \mathcal{D}(X) \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{A}(X \backslash H) & \xrightarrow{\mathcal{P}} & \mathcal{D}(X \backslash H)
\end{array}$$

commutes; more generally, when Y is transversal to H, the diagram

$$\begin{array}{ccc}
\mathcal{A}(X)_H & \xrightarrow{\mathcal{P}} & \mathcal{D}(X)_Y \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{A}(X \backslash H) & \xrightarrow{\mathcal{P}} & \mathcal{D}(X \backslash H)_Y
\end{array}$$

commutes. The following diagram commutes, part of which are the squares just mentioned:

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[rr, "\mathcal{P}"] \arrow[dr] \arrow[dd] & & \mathcal{D}(X) \arrow[dd] & \\
& \mathcal{A}(X)_H \arrow[rr, "\mathcal{P}"] \arrow[dd] & & \mathcal{D}(X)_Y \arrow[ul] \arrow[dd] \\
\mathcal{A}(X \backslash H) \arrow[rr, "\mathcal{P}"] \arrow[dr, "id"'] & & \mathcal{D}(X \backslash H) & \\
& \mathcal{A}(X \backslash H) \arrow[rr, "\mathcal{P}"] & & \mathcal{D}(X \backslash H)_Y\,. \arrow[ul]
\end{tikzcd}$$

Also we have a commutative diagram

$$
\begin{tikzcd}
\mathcal{A}(X|Y)_H \arrow[rr, "\mathcal{P}"] \arrow[dr] \arrow[dd] & & \mathcal{D}(X|Y) \arrow[dr] \arrow[dd] & \\
& \mathcal{A}(X\backslash H|Y) \arrow[rr, "\mathcal{P}"] \arrow[dd] & & \mathcal{D}(X\backslash H|Y) \arrow[dd] \\
\mathcal{A}(X)_H \arrow[rr, "\mathcal{P}"] \arrow[dr] & & \mathcal{D}(X)_Y \arrow[dr] & \\
& \mathcal{A}(X\backslash H) \arrow[rr, "\mathcal{P}"] & & \mathcal{D}(X\backslash H)_Y \,.
\end{tikzcd}
$$

(3.11) **Proposition.** *The maps (3.8.1), (3.8.2) and (3.8.3) are quasi-isomorphisms.*

(3.12) Composing the map $s : \mathcal{A}(X)\langle H\rangle \to \mathcal{A}(X\backslash H)$ with $\mathcal{P} : \mathcal{A}(X\backslash H) \to \mathcal{D}(X\backslash H)$, we get a quasi-isomorphism

$$\mathcal{P} : \mathcal{A}(X)\langle H\rangle \to \mathcal{D}(X\backslash H)\,.$$

More generally we have $\mathcal{P} : \mathcal{A}(X)\langle H\rangle \to \mathcal{D}(X\backslash H)_Y$ if H and Y are transversal.

Also, composing $s : \mathcal{A}(X|Y)\langle H\rangle \to \mathcal{A}(X\backslash H|Y)$ we have a quasi-isomorphism

$$\mathcal{P} : \mathcal{A}(X|Y)\langle H\rangle \to \mathcal{D}(X\backslash H|Y)\,.$$

We now define the homological Hodge complex of $(X\backslash Y|H)$.

(3.13) We have a triple of complexes

$$\mathcal{C}(X\backslash Y|H) \xrightarrow{\Phi} \mathcal{D}(X\backslash Y|H) \xleftarrow{\mathcal{P}} \mathcal{A}(X|H)\langle Y\rangle \tag{3.13.1}$$

When H is empty, we each of the three complexes is equipped with the weight filtration W, and third complex with additional filtration F, and the triple gives a mixed Hodge complexes of sheaves on X, see Part II.

In general, viewing the complex $\mathcal{C}(X\backslash Y|H)$ as a complex of complexes

$$0 \to \mathcal{C}(X\backslash Y) \to \mathcal{C}(H^{(1))}\backslash H^{(1))} \cap Y) \to \cdots$$

one has an induced filtration W on the total complex. Similarly the complexes $\mathcal{D}(X\backslash Y|H)$ and $\mathcal{A}(X|H)\langle Y\rangle$ have the filtration W. In addition, $\mathcal{A}(X|H)\langle Y\rangle$ has the Hodge filtration F as well.

Thus the triple (3.13.1) gives a mixed Hodge complex, which will be denoted by $\mathbb{L}(X\backslash Y|H)$. We call this the homological Hodge complex for $(X\backslash Y|H)$.

The next result follows from the case H empty, which was shown in Part II.

(3.14) **Theorem.** *The mixed Hodge complex $\mathbb{L}(X\backslash Y|H)$ is isomorphic to the mixed Hodge complex given by Deligne-Beilinson.*

References

[Be] Beilinson, A., Notes on absolute Hodge cohomology, Contemp. Math., 55 American Mathematical Society, Providence, RI, 1986, 35-68.

[Br] Bredon, G. E. : Sheaf theory. Second edition. Graduate Texts in Mathematics, 170. Springer-Verlag, New York, 1997.

[De] Deligne, P.: Deligne, P., Théorie de Hodge. II and III, Inst. Hautes Études Sci. Publ. Math. 40(1971), 5-57; ibd. 44(1974), 5-77.

[Go] Godement, R. : Topologie algébrique et Théorie des faisceaux. Hermann, 1958.