Homological Hodge complexes II

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We construct the *homological* Hodge complex of a smooth complex variety, and verify that the existence of a comparison isomorphism to the standard (or the cohomological) Hodge complex.

1 Filtered complexes and Hodge complexes

We review the notion of (mixed) Hodge complexes from [De] and [Be].

(1.1) In what follows a complex are always assumed to be bounded below.

Consider a diagram of the form

$$(K^\bullet_{\mathbb{Q}}, W) \xrightarrow{a} (K'^\bullet_{\mathbb{C}}, W) \xleftarrow{b} (K^\bullet_{\mathbb{C}}, W, F)\,, \tag{1.1.1}$$

where

- $K^\bullet_{\mathbb{Q}}$ is a complex of $\mathbb{Q}$-vector spaces with a (finite) increasing filtration $W_\bullet$, $K'^\bullet_{\mathbb{C}}$ is a complex of $\mathbb{C}$-vector spaces with an increasing filtration $W_\bullet$, $K^\bullet_{\mathbb{C}}$ is a complex of $\mathbb{C}$-vector spaces with two filtrations W, F (W increasing and F decreasing),
- a is a map of complexes that induces a filtered quasi-isomorphism $(K^\bullet_{\mathbb{Q}}, W)\otimes\mathbb{C} \to (K'^\bullet_{\mathbb{C}}, W)$, and b gives a filtered quasi-isomorphism $(K^\bullet_{\mathbb{C}}, W) \to (K'^\bullet_{\mathbb{C}}, W)$.

Such a diagram K will be called a *triple of filtered complexes* of vector spaces.

Definition. We call K a $\mathbb{Q}$-*Hodge complex* if in addition the following conditions are satisfied:

(i) $H^i(K^\bullet_{\mathbb{Q}})$ are finite dimensional $\mathbb{Q}$-vector spaces.

(ii) For any a, the differential d of the filtered complex $(\mathrm{Gr}^W_a K_{\mathbb{C}}, F)$ is strictly compatible with the filtration F.

(iii) The isomorphism $H^i(\mathrm{Gr}^W_a K_{\mathbb{Q}}) \otimes \mathbb{C} \cong H^i(\mathrm{Gr}^W_a K_{\mathbb{C}})$ induced by the diagram and the filtration F on $H^i(\mathrm{Gr}^W_a K_{\mathbb{C}})$ gives $H^i(\mathrm{Gr}^W_a K_{\mathbb{Q}})$ the structure of a pure $\mathbb{Q}$-Hodge structure of weight a.

This notion (indeed for a more general coeffficient) is due to Beilinson [Be]. It is known that, if K is a $\mathbb{Q}$-Hodge complex then the filtered vector spaces $(H^i(K_{\mathbb{Q}}), W)$, $(H^i(K_{\mathbb{C}}), W, F)$ and the filtered isomorphism $(H^i(K_{\mathbb{Q}}), W) \otimes \mathbb{C} \cong (H^i(K_{\mathbb{C}}), W)$ define on $H^i(K_{\mathbb{Q}})$ a $\mathbb{Q}$-mixed Hodge structure (by an argument parallel to [De], II, (3.2.10) and III, (8.1.9)).

(1.2) A triple of filtered complexes (1.1.1) satisfying the same conditions (i), (ii) as above and, in place of (iii), condition (iii)$'$ below, will be called a $\mathbb{Q}$-*mixed Hodge complex*.

(iii)$'$ The isomorphism $H^i(\mathrm{Gr}_a^W K_{\mathbb{Q}}) \otimes \mathbb{C} \cong H^i(\mathrm{Gr}_a^W K_{\mathbb{C}})$ given by the diagram and the filtration F on $H^i(\mathrm{Gr}_a^W K_{\mathbb{C}})$ is a pure $\mathbb{Q}$-Hodge structure of weight $i + a$.

If K is a $\mathbb{Q}$-mixed Hodge complex, by taking décalage with respect to the filtration W we obtain another triple of filtered complexes

$$(K^\bullet_{\mathbb{Q}}, \mathrm{Dec}(W)) \xrightarrow{a} (K'^\bullet_{\mathbb{C}}, \mathrm{Dec}(W)) \xleftarrow{b} (K^\bullet_{\mathbb{C}}, \mathrm{Dec}(W), F)\,;$$

this is a $\mathbb{Q}$-Hodge complex.

Remark. 1. Our notion of mixed Hodge complex differs from Deligne's [De], III. A mixed Hodge complex in [De] consists a pair of objects in filtered derived categories, together with a comparison isomorpism in the (filtered) derived category. Of course precedes Deligne's notion precedes [Be].

A $\mathbb{Q}$-mixed Hodge complex in our sense clearly gives rise to a mixed Hodge complex in the sense of Deligne.

2. In the literature more general notion of A-(mixed) Hodge complexes, for a coefficient ring A, are considered; the formulations need to be changed in the obvious manner. We restrict ourselves to the case $A = \mathbb{Q}$.

(1.3) For $\mathbb{Q}$-Hodge complexes one can construct the associated derived category as follows. For details see [Be].

Let $\mathcal{C}^+_{\mathcal{H}}$ denote the category of $\mathbb{Q}$-Hodge complexes. A morphism of $\mathbb{Q}$-Hodge complexes is a triple of filtered complexes making the obvious diagram commutative. One has the notion of homotopy between morphisms, thus one can form the homotopy category $\mathcal{K}^+_{\mathcal{H}}$. There is the cohomology functor $H^\bullet : \mathcal{K}^+_{\mathcal{H}} \to (\mathbb{Q} - \mathrm{Vect})$ to the category of $\mathbb{Q}$-vector spaces. Via the cohomology functor one has the class of quasi-isomorphisms. By inverting them one arrives at a triangulated category $\mathcal{D}^+_{\mathcal{H}}$, which may be called the *derived category of* $\mathbb{Q}$*-Hodge complexes*.

In this paper, we will have $\mathbb{Q}$-mixed Hodge complexes K, which we turn to $\mathbb{Q}$-Hodge complexes, then view them as objects in the category $\mathcal{D}^+_{\mathcal{H}}$.

The $\mathbb{Q}$-mixed Hodge complexes K we encounter will come from $\mathbb{Q}$-mixed Hodge complexes of sheaves, to be explained below.

(1.4) Consider now sheaves on a topological space X.

A *triple of filtered complexes of sheaves* is a diagram

$$(\mathcal{K}^\bullet_{\mathbb{Q}}, W) \xrightarrow{a} (\mathcal{K}'^\bullet_{\mathbb{C}}, W) \xleftarrow{b} (\mathcal{K}^\bullet_{\mathbb{C}}, W, F)\,. \tag{1.4.1}$$

Here

- $(\mathcal{K}_{\mathbb{Q}}^{\bullet}, W)$ is a complex of $\mathbb{Q}$-sheaves with a (finite) increasing filtration $W_{\bullet}$, $(\mathcal{K'}_{\mathbb{C}}^{\bullet}, W)$ is a complex of $\mathbb{C}$-sheaves with an increasing filtration $W_{\bullet}$, and $(\mathcal{K}_{\mathbb{C}}^{\bullet}, W, F)$ is a complex of $\mathbb{C}$-sheaves with an increasing filtration $W_{\bullet}$ and a decreasing filtration F;

- the arrow a is a morphism of filtered complexes $(\mathcal{K}_{\mathbb{Q}}^{\bullet}, W) \to (\mathcal{K'}_{\mathbb{C}}^{\bullet}, W)$ which induces a filtered quasi-isomorphism $(\mathcal{K}_{\mathbb{Q}}^{\bullet}, W) \otimes \mathbb{C} \to (\mathcal{K'}_{\mathbb{C}}^{\bullet}, W)$, and b is a filtered quasi-isomorphism $(\mathcal{K}_{\mathbb{C}}^{\bullet}, W) \to (\mathcal{K'}_{\mathbb{C}}^{\bullet}, W)$.

Definition. The filtered triple (1.4.1) is said to be a $\mathbb{Q}$-*Hodge complex of sheaves* (resp. $\mathbb{Q}$-*mixed Hodge complex of sheaves*) if in addition the following condition holds:

(i) For each term $\mathcal{K}_{\mathbb{Q}}^{i}$ of the complex $\mathcal{K}_{\mathbb{Q}}$, the graded object $\mathrm{Gr}^{W}(\mathcal{K}_{\mathbb{Q}}^{i})$ is Γ-acyclic, where $\Gamma = \Gamma(X, -)$; similarly for $\mathcal{K'}_{\mathbb{C}}$ the object $\mathrm{Gr}^{W}(\mathcal{K}_{\mathbb{Q}}^{i})$ is Γ-acyclic; and for $\mathcal{K}_{\mathbb{C}}^{\bullet}$, the object $\mathrm{Gr}^{W}\mathrm{Gr}^{F}(\mathcal{K}_{\mathbb{C}}^{i})$ is Γ-acyclic.

(ii) By the first condition, application of the functor $\Gamma(X, -)$ yields a triple of filtered complexes of vector spaces. It is a $\mathbb{Q}$-Hodge complex.

A $\mathbb{Q}$-mixed Hodge complex of sheaves gives a $\mathbb{Q}$-Hodge complex of sheaves by taking décalage for W.

A $\mathbb{Q}$-mixed Hodge complex of sheaves in our sense gives, by passing to the derived category, a cohomological $\mathbb{Q}$-mixed Hodge complex as defined in [De], (8.1.6).

2 Push-out of complexes and functorial properties

We collect the basic properties of push-out of complexes, to be used later.

(2.1) Given maps of complexes $u : K \to L$ and $v : K \to M$, consider the map $(u, v) : K \to L \oplus M$, take its cone: $Q = \mathrm{Cone}(K \to L \oplus M)$:

$$\begin{array}{ccc} K & \xrightarrow{u} & L \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle v'} \\ M & \xrightarrow{u'} & Q \end{array}$$

There are canonical maps u' and v' as indicated. Q is called the *push-out* of $(K; u, v)$. Recall that $Q^p = K^{p+1} \oplus L^p \oplus M^p$, and the differential sends $(x; y, z) \in Q^p$ to $(-dx; u(x)+dy, v(x)+dz)$. We will also say that $M \to Q$ is obtained from $K \to L$ by pushing out along v.

If we define a map of degree -1, $S : K^{\bullet} \to Q^{\bullet -1}$ by $S(x) = (x; 0, 0)$ then we have the identity $dS + Sd = v'u + u'v$.

(2.1.1) The push-out square satisfies the following universal property. Assume given a square of complexes

$$\begin{array}{ccc} K & \xrightarrow{u} & L \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle f} \\ M & \xrightarrow{g} & R \end{array}$$

and a map $U : K^\bullet \to R^{\bullet -1}$ such that $dU + Ud = fu + gv$. There is then a unique map of complexes $q : Q \to R$ such that

$$qu' = g\,, \quad qv' = f\,, \quad \text{and} \quad qS = T\,.$$

(2.1.2) The push-out has the obvious functoriality: Suppose we have another pair of maps of complexes $u_1 : K_1 \to L_1$ and $v_1 : K_1 \to M_1$; also given are maps of complexes $k : K \to K_1$, $\ell : L \to L_1$, and $m : M \to M_1$, satisfying $\ell u = u_1 k$ and $mv = v_1 k$. Let Q_1 be the push-out of $(K_1; u_1, v_1)$ with maps $v_1' : L_1 \to Q_1$, $u_1' : M_1 \to Q_1$ and degree -1 map $S_1 : K_1 \to Q_1$. Then there is a unique map of complexes $q : Q \to Q_1$ such that $qu' = mu_1'$, $qv' = \ell v_1'$, and $qS = S_1 q$.

(2.2) We formulate more of the functorial properties. Suppose we are given

(i) maps of complexes $u : K \to L$, $v : K \to M$, $u_1 : K_1 \to L_1$ and $v_1 : K_1 \to M_1$,

(ii) maps of complexes $k : K \to K_1$, $\ell : L \to L_1$, and $m : M \to M_1$,

(iii) maps of degree -1, $T : K \to L_1$ and $T' : K \to M_1$ satisfying the identities

$$dT + Td = \ell u - u_1 k\,, \qquad dT' + T'd = mv - v_1 k\,,$$

and

(iv) another complex R together with maps of complexes $v_1' : L_1 \to R$ and $u_1' : M_1 \to R$, and a map $U : K_1 \to R$ of degree -1 satisfying

$$dU + Ud = v_1' u_1 + u_1' v_1\,.$$

Let Q be the push-out of $(K; u, v)$ as in the previous paragraph. Then there exists a unique map of complexes $q : Q \to R$ such that the following identities hold:

$$qu' = u_1' m\,, \quad qv' = v_1' \ell \quad \text{and} \quad qS = Uk + v_1' T + u_1' T'\,.$$

$$\begin{tikzcd}
K \arrow[rr, "u"] \arrow[dr, "v"] \arrow[dd, "k"'] & & L \arrow[dr, "v'"] \arrow[dd, "\ell"] & \\
& M \arrow[rr, "u'"] \arrow[dd, "m"] & & Q \arrow[dd, "q"] \\
K_1 \arrow[rr, "u_1"] \arrow[dr, "v_1"'] & & L_1 \arrow[dr, "v_1'"] & \\
& M_1 \arrow[rr, "u_1'"] & & R
\end{tikzcd}$$

The verification is straightforward. Note the special cases:

(1) The universal property (2.1.1) is the case where $k : K \to K_1$, $\ell : L \to L_1$, and $m : M \to M_1$ are the identity maps and $T = T' = 0$.

(2) If $R = Q_1$ is the push-out of $(K_1; u_1, v_1)$ and S_1 is the associated homotopy, then there exists an induced map of complexes $q : Q \to Q_1$ that satisfy the mentioned conditions with $U = S_1$. Note that if further $T = T' = 0$, then it reduces to the obvious case (2.1.2).

(2.3) The following properties are easily shown.

For the push-out (2.1), if u is a quasi-isomorphism, then so is u'.

In (2.2), if k, ℓ, m are quasi-isomorphisms, and $R = Q_1$, then q is also a quasi-isomorphism.

3 Semi-analytic triangulations and the semi-analytic chain complex

In what follows simplicial complex means a geometric, locally finite simplicial complex in an $\mathbb{R}^N$, as in [Mu], §2. Thus a simplicial complex K is set of simplices σ, satisfying certain conditions. The polytope of K, denoted by $|K|$, is the union of the simplices in K. The relative interior of a simplex σ will be denoted by $\overset{\circ}{\sigma}$. Let $C_*(K)$ be the complex of oriented simplices of K, with coefficients in $\mathbb{Z}$ (see [Mu], §5). It is known that the homology of this complex is canonically isomorphic to the Borel-Moore homology $H_*(|K|) = H_*(|K|;\mathbb{Z})$ of $|K|$, (cf. [Br], [Ha]). One can also take $\mathbb{Q}$ as the coefficient ring; from §2 on we will always do so.

(3.1) We recall a theorem in [Ło]. Let M a real analytic manifold satisfying the second axiom of countability. By [Ło], Theorem 2, there exist a locally finite simplicial complex K and a homeomorphism $h : |K| \to M$ satisfiying the following condition:

(i) For each simplex σ in K, $h(\overset{\circ}{\sigma})$ is a semi-analytic subset as well as an analytic submanifold of M, and the map $\overset{\circ}{\sigma} \to h(\overset{\circ}{\sigma})$ is an analytic isomorphism.

Given a locally finite collection of semi-analytic subsets $\{B_\nu\}$ of M, one may arrange that we also have:

(ii) Each B_ν is a union of some of the sets $h(\overset{\circ}{\sigma})$, for σ a simplex of K. (We will then say that the collection $\{h(\overset{\circ}{\sigma})\}$ is compatible with (B_ν).)

Such a pair $(K; h : |K| \to M)$ is called a *semi-analytic trialgulation* of M. (When it does not cause confusion, we will simply refer to the map h for a triangulation.) If it also satisfies (ii), then the triangulation is *compatible* with $\{B_\nu\}$.

(3.2) Let K and L be simplicial complexes. Suppose $f : |L| \to |K|$ is a homeomorphism satisfying the following condition:

Each simplex σ in K is a finite union of $f(\tau)$, for some simplices τ of L.

We then say that f is a *subdivision* of simplicial complexes.

Remark. (1) An equivalent condition is that for each simplex σ of K, its interior $\overset{\circ}{\sigma}$ is a finite union of $f(\overset{\circ}{\tau})$ for some simplices τ of L.

(2) This is a generalization of subdivision of K in the sense of [Mu]. If K and L are simplicial complexes of the same $\mathbb{R}^N$ and $|K| = |L|$, then $id : |L| \to |K|$ is a subdivision in our sense iff L is a subdivision of K in the sense of [Mu].

For a simplex σ in K, the set

$$L(\sigma) = \{\tau \in L \mid h(\tau) \subset \sigma\}$$

is a subcomplex of L. Since $|L(\sigma)| = h^{-1}(\sigma)$, which is homeomorphic to σ, $L(\sigma)$ is an acyclic simplicial complex. The proof of the following result is parallel to that of [Mu], §17.

(3.3) **Proposition.** *Given a subdivision of simplicial complexes f as above, there exists a unique augmentation preserving chain map*

$$\lambda : C_*(K) \to C_*(L)$$

such that $\lambda(\sigma)$ is carried by $L(\sigma)$ for each simplex σ of K. It is a quasi-isomorphism.

The assignment of λ to f is contravariantly functorial.

We give here a direct definition of the map λ. For each oriented p-simplex σ of K, we have

$$\sigma = \cup f(\sigma')$$

a finite union of p-simplices σ' of L. Orient each σ' such that the induced orientation (by the homeomorphism f) on $f(\sigma')$ is compatible with the orientation of σ. Let

$$\lambda(\sigma) = \sum \sigma'$$

and extend it by linearity to define a map λ. One verifies easily that this gives an augmentation-preserving chain map, and clearly carried by $L(\sigma)$ by definition, thus it coincides with the map λ in the theorem.

(3.4) Let $(K; h : |K| \to M)$ and $(L; h' : |L| \to M)$ be semi-analytic triangulations of M. A *morphism of semi-analytic triangulations* from $(L; h')$ to $(K; h)$ is a subdivision $f : |L| \to |K|$ satisfying $h \circ f = h'$.

Note first that there is at most one morphism f between two triangulations, which is given as $f = h^{-1} \circ h'$.

Second, if f is a morphism of semi-analytic triangulations, and if τ is a simplex of L, σ is K such that $f(\mathring{\tau}) \subset \mathring{\sigma}$, then $f(\mathring{\tau})$ is a semi-analytic subset as well as an analytic submanifold of $\mathring{\sigma}$.

We consider the category of triangulations of M, and denote it by $Tr(M)$. The objects are the semi-analytic triangulations of A, $(K; h : |K| \to A)$. The arrows are the morphisms defined above.

When we are given a locally finite collection of semi-analytic subsets $\mathcal{B} = \{B_\nu\}$, we may consider the full subcategory $Tr(M; \mathcal{B})$ of $Tr(M)$, consisting of those triangulations compatible with $\mathcal{B}$.

(3.5) **Proposition.** *The category of semi-analytic triangulations $Tr(M)$ is cofiltered. The same holds for $Tr(M; \{B_\nu\})$, given a locally finite collection $\{B_\nu\}$. Further, the subcategory $Tr(M; \{B_\nu\})$ is cofinal in $Tr(M)$.*

Proof. Given two triangulations $h : |K| \to M$ and $h' : |L| \to M$, consider the collection of semi-analytic sets in $|K|$,

$$\mathcal{B} = \{h(\overset{\circ}{\sigma}) \cap h'(\overset{\circ}{\sigma'}) \mid \sigma \in K, \quad \sigma' \in L\},$$

and apply (3.1). There is a simplicial complex L and a homeomorphism $g : |L| \to M$ such that the collection $\{g(\overset{\circ}{\nu})\}$ (for ν simplices of L) is compatible with $\mathcal{B}$. If $f : |L| \to |K|$ and $f' : |L| \to |K'|$ are homeomorphisms such that $g = h \circ f = h' \circ f'$, then one sees that $\{f(\overset{\circ}{\nu})\}$ is compatible with $\{\overset{\circ}{\sigma}\}$, and $\{f'(\overset{\circ}{\nu})\}$ is compatible with $\{\overset{\circ}{\sigma'}\}$; then f and f' are morphisms of triangulations. This shows the connectivity of the category $Tr(M)$.

Suppose that $f : |L| \to |K|$ and $f' : |L'| \to |K|$ are morphisms of triangulations of M. Considering the collection of subsets

$$\{f(\overset{\circ}{\tau}) \cap f'(\overset{\circ}{\tau'}) \mid \tau \in L, \quad \tau' \in L'\}$$

and using the same theorem, we find another triagulation $|L''| \to M$ and morphisms of triangulations $g : |L''| \to |L|$ and $g' : |L''| \to |L|$ so that $g \circ f = g' \circ f'$.

Also it is clear that if f and f' are morphisms of triangulations $|L| \to |K|$, then $f = f'$. Thus the category is cofiltered.

The argument for $Tr(M; \{B_\nu\})$ is an obvious modification. That it is cofinal in $Tr(M)$ follows easily from (3.1). □

(3.6) We have a contravariant functor from $Tr(M)$ to the category of chain complexes, which assigns to a triangulation $(K; h : |K| \to M)$ the simplicial chain complex $C_*(K)$ and assigns to each morphism $f : (L; h') \to (K; h)$ the homomorphism $\lambda : C_*(K) \to C_*(L)$.

One can form the inductive limit over the cofiltered category $Tr(M)$, and obtain a chain complex

$$\varinjlim C_*(K);$$

we denote it by $C_*(M)$. This is the complex of *semi-analytic chains* in M.

Given a locally finite collection $\mathcal{B} = \{B_\nu\}$, the same inductive limit over the subcategory $Tr(M; \mathcal{B})$ gives us a quasi-isomorphic subcomplex, denoted by $C_*(M; \mathcal{B})$. There is an inclusion $C_*(M; \mathcal{B}) \to C_*(M)$, and it is a quasi-isomorphism.

(3.7) Let X be a closed semi-analytic subset of M. Let h be an object of $(M; \{X\})$, namely an analytic triangulation $h : |K| \to M$ compatible with X.

Then the subset of K given by

$$K_X := \{\sigma \in K \mid h(\sigma) \subset X\}$$

is a subcomplex of K, and h restricts to a homeomorphism

$$h_X : |K_X| \to X;$$

this is the induced triangulation of X from $(K; h)$.

There is a natural inclusion $C_*(K_X) \to C_*(K)$. The next proposition is obvious.

(3.8) **Proposition.** *Suppose $(K; h)$ is an analytic triangulation of $(M; \{X\})$. If $f : (L, h') \to (K, h)$ is a morphism in $Tr(M)$, then (L, h') is also an analytic triangulation of $(M; \{X\})$. Hence one has a subcomplex L_X of L and f restricts to a morphism of triangulations $f_X : L_X \to K_X$ over X.*

Consequently there is a commutative diagram of complexes

$$\begin{array}{ccc} C_*(K_X) & \xrightarrow{\lambda} & C_*(L_X) \\ \downarrow & & \downarrow \\ C_*(K) & \xrightarrow{\lambda} & C_*(L)\,. \end{array}$$

(3.9) We define

$$C_*(X) = \varinjlim C_*(K_X)\,, \qquad \text{limit over the category } Tr(M; \{X\}).$$

More generally if $\mathcal{B}$ contains X as a member, we have

$$C_*(X; \mathcal{B}) := \varinjlim C_*(K_X)\,, \qquad \text{limit over } Tr(M; \mathcal{B}).$$

(More precise notation would be $C_*(X; M)$ and $C_*(X; (M, \mathcal{B}))$, respectively.) There is an inclusion $C_*(X; \mathcal{B}) \to C_*(X)$, which is a quasi-isomorphism.

(3.10) Let U be an open set of M. Consider a semi-analytic triangulation $h : |K| \to M$ of M, and a semi-analytic triangulation $(K_U; h_U : |K_U| \to U)$ of U. Let $j : |K_U| \to |K|$ be the map making the diagram

$$\begin{array}{ccc} |K| & \xrightarrow{h} & M \\ {\scriptstyle j}\uparrow & & \cup \\ |K_U| & \xrightarrow{h_U} & U\;. \end{array}$$

commute; it is a homeomorphism to the open set $h^{-1}(U)$. The triangulation h_U is said to be *subordinate to* h if for any simplex τ of K_U, there exists a simplex σ of K such that $j(\tau) \subset \sigma$.

For any semi-analytic triangulation $h : |K| \to M$ of M and for any open set U, there is a semi-analytic triangulation of U that subordinate to h. Indeed, if K is in $\mathbb{R}^N$, one may choose K_U to be a simplicial complex in the same $\mathbb{R}^N$ in such a way that $|K_U|$ is a subset of K, and that each simplex of K_U is an affine subset of a simplex of K. However we will need the more general notion we introduced above.

If $h_U : |K_U| \to U$ is compatible with h, there is an induced map of complexes $\rho : C_*(K) \to C_*(K_U)$ called restriction, defined as follows.

For each oriented p-simplex σ of K, write $\sigma \cap U$ as the union of p-simplices σ' of K_U that are contained in σ, so that $\sigma \cap U = \cup \sigma'$. Give each σ' the orientation compatible with that for

σ, and let $\rho(\sigma) = \sum \sigma'$. Extending by linearity, one obtains the map ρ, and one verifies it is a map of complexes. Note that $|\rho(\alpha)| = |\alpha| \cap U$ for any chain α in $C_*(K)$ (recall $|\alpha|$ denotes the support of a chain).

(3.11) **Proposition.** *Suppose $(K; h)$ and $(L; h')$ are semi-analytic triangulations of an analytic manifold M, and $f : |L| \to |K|$ is a morphism of triangulations. Suppose also U is an open set of M, $h_U : |K_U| \to U$ is a semi-analytic triangulation subordinate to the triangulation h. Then there exists a semi-analytic triangulation $h'_U : |L_U| \to U$ subordinate to h', and a morphism of triangulations $f_U : |L_U| \to |K_U|$ from h'_U to h_U such that the diagram*

$$\begin{array}{ccc} |L| & \xrightarrow{f} & |K| \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle j} \\ |L_U| & \xrightarrow{f_U} & |K_U| \ . \end{array}$$

commutes.

Proof. Consider the collection of semi-analytic sets of U,

$$\mathcal{B} = \{h'(\mathring{\tau}) \cap h_U(\mathring{\sigma}) \mid \tau \in L, \quad \sigma \in K_U\} \, .$$

Choose a semi-analytic triangulation $h'_U : |L_U| \to U$ that is compatible with $\mathcal{B}$. We then have the required properties. ☐

(3.12) Under the assumption of Proposition (3.11) we have a commutative diagram of complexes

$$\begin{array}{ccc} C_*(K) & \xrightarrow{\lambda} & C_*(L) \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ C_*(K_U) & \xrightarrow{\lambda} & C_*(L_U) \end{array}$$

where the horizontal maps are the subdivisions operators and the vertical ones the restrictions. Passing to the limit over the triangulations of M, we get a map of complexes

$$C_*(M) \to C_*(U)$$

which we call restriction. The image of α will be written $\alpha|_U$.

If U and V are open sets of M, arguing as above with M replaced with U, we have restriction $C_*(U) \to C_*(V)$. This gives us a complex of presheaves on M; we will denote it by $\mathcal{C}_{M\,*}$ or $\mathcal{C}_*(M)$.

(3.13) A chain α in $C_i(M)$ determines the subsets $A = |\alpha|$ and $B = |\partial\alpha|$ of M; both are semi-analytic subsets of M, $A \supset B$, and $\dim A = i$, $\dim B = i - 1$ (unless empty). Also α determines a class $c(\xi) \in H_i(A, B)$ in the Borel-Moore homology of the pair (A, B). There is a canonical isomorphism $H_i(A, B) \cong H_i(A \backslash B)$.

Conversely if A, B are closed semi-analytic subsets of M of dimension i and $i-1$, respectively (unless empty), and $c \in H_i(A, B)$ is a class, there corresponds a chain $\alpha \in C_i(M)$ such that $|\alpha| \subset A$, $|\partial\alpha| \subset B$, and such that the image of $c(\xi) \in H_i(|\alpha|, |\partial\alpha|)$ equals c. For this one applies (3.1) to M and the collection $\{A, B\}$ to find a triangulation which supports c.

Using this viewpoint, one can show that the presheaf $U \mapsto C_i(U)$ is sheaf. Indeed suppose U is an open set of M, $U = \cup U_\alpha$ is its open covering, and ξ_α be elements of $C_i(U_\alpha)$ that satisfy the compatibility condition. Let $A_\alpha = |\xi_\alpha|$, $B_\alpha = |\partial(\xi_\alpha)|$ which are semi-analytic subsets of U_α. Then (A_α) (resp. (B_α)) glue to define a semi-analytic set of M of dimension i and $i-1$, respectively. Let $\mathcal{H}_i$ be the homology sheaf $V \mapsto H_i(V)$ on $A \backslash B$. Then the sections ξ_α on $A_\alpha \backslash B_\alpha$ glue to a section $\xi \in \Gamma(A\backslash B, \mathcal{H}_i) = H_i(A\backslash B)$. It determines an element $\alpha \in C_i(U)$ that restrict to ξ_α on U_α.

This we call the complex of sheaves of semi-analytic chains on M, and denote it by $\mathcal{C}_{M,*}$ or $\mathcal{C}_*(M)$.

(3.14) If $h : |K| \to M$ is a semi-analytic triangulation of M, there is an injective map of complexes

$$C_*(K) \to \mathcal{S}_*^{BH}(M)$$

where the right hand side is the complex of sheaves of semi-analytic chains as introduced by Bloom-Herrera. (See [B-H], §2.7.) This induces an isomorphism of complexes $C_*(M) \to \mathcal{S}_*^{BH}(M)$. This being the case for each open set of M, we have an isomorphism of complexes of sheaves $C_*(U) \to \mathcal{S}_{M,*}^{BH}(U)$.

We refer to [Br], Chap. V for the notion of cosheaves and corresolutions. The following theorem is proved using Theorem 12.20 of *loc. cit.*.

(3.15) **Theorem.** *The complex $\mathcal{C}_{X,*}$ consists of c-soft sheaves. The associated complex of cosheaves $U \mapsto \Gamma_c(U, \mathcal{C}_{X,*})$ has an augmentation to $\mathbb{Z}_X$, which makes it a quasi-corresolution. In particular the cohomology of $\Gamma(X, \mathcal{C}_{X,*})$ is canonically isomorphic to the Borel-Moore homology of X.*

(3.16) Generalizing (3.10) we verify the following.

Suppose U is an open set of M, X is a closed semi-analytic set of M, and $V = X \cap U$. Suppose an object $(K; h)$ of $Tr(M; \{X\})$ is given; let h_X be the induced triangulation of X. Then there exists an object $(K_U; h_U)$ of $Tr(U; \{V\})$ such that the following conditions hold:

- $(K_U; h_U)$ is subordinate to $(K; h)$, and
- $(K_V; h_V)$ is subordinate to $(K_X; h_X)$. (Here $(K_V; h_V)$ is the triangulation of V induced from $(K_U; h_U)$).

Consequently we have a commutative diagram of complexes

$$\begin{array}{ccc} C_*(K_X) & \longrightarrow & C_*(K) \\ \rho\big\downarrow & & \big\downarrow\rho \\ C_*(K_V) & \longrightarrow & C_*(K_U) \end{array}$$

(3.17) Generalizing Proposition (3.11), we can show:

Suppose given a morphism $f : (L; h') \to (K; h)$ in $Tr(M; \{X\})$. Also given an open set U of M, and an object $(K_U; h_U)$ of $Tr(U)$ that is subordinate to $(K; h)$. Then there exists an object $(L_U; h'_U)$ of $Tr(U; \{V\})$ such that

$$(L_U; h'_U) \text{ is subordinate to } (L; h'), \quad \text{and} \quad (L_V; h'_V) \text{ is subordinate to } (L_X; h'_X).$$

We have thus a commutative diagram of complexes

$$\begin{array}{ccc} C_*(K_X) & \xrightarrow{\lambda} & C_*(L_X) \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle\rho} \\ C_*(K_V) & \xrightarrow{\lambda} & C_*(L_V) \end{array}$$

which is compatible with the commutative diagram in (3.8). It induces a map $\rho : C_*(X) \to C_*(V)$, giving us a complex of sheaves $\mathcal{C}_{X,*}$.

(3.18) Let M be an analytic manifold. For any element $\alpha \in C_*(M)$ there exists an analytic triangulation $h : |K| \to M$ such that $\alpha \in C_*(K)$ (obvious from the definitions).

It follows from this and the previous subsection that, if X is a closed semi-analytic subset of M, then one has

$$C_*(X) = \{\alpha \in C_*(M) \mid |\alpha| \subset X\}.$$

If $i : X \to M$ denotes the inclusion, one has $i^! \mathcal{C}_{M,*} = \mathcal{C}_{X,*}$.

Let $j : U = M - X \to M$ be the inclusion of the open set. We have a quasi-isomorphism

$$\mathcal{C}_M / \mathcal{C}_X \to j_* \mathcal{C}_U.$$

4 Complexes of forms and currents

Let X be a smooth connected complex variety of dimension n. We will consider sheaves of Λ-vector spaces on a smooth complex variety, where $\Lambda = \mathbb{Q}$ or $\mathbb{C}$; a sheaf is denoted by $\mathcal{F}$. Also consider complexes of such sheaves $\mathcal{F}^\bullet$, and filtered complexes of sheaves $(\mathcal{F}^\bullet, W)$ where W is a filtration by subcomplexes. We always assume that a filtration is finite, namely it is a finite filtration on each component.

One may apply the global section functor to get a filtered complex of Λ-vector spaces $\Gamma(X, (\mathcal{F}^\bullet, W)\,)$; similarly for the direct image f_* for a map $f : X \to X'$, or for any other left exact functor.

One has the derived functor on the filtered derived category of sheaves. If $(\mathcal{F}^\bullet, W)$ is a filtered complex (of sheaves) and $(\mathcal{F}^\bullet, W) \to (\mathcal{F}'^\bullet, W)$ is a filtered quasi-isomorphism such that the graded pieces of $\mathcal{F}'$ are Γ-acyclic, then one has $\mathbb{R}\Gamma(X, (\mathcal{F}^\bullet, W)\,) = \Gamma(X, (\mathcal{F}'^\bullet, W)\,)$. The same applies to the direct image f_* and its derived functor.

(4.1) In the rest of this paper we work mainly under the following assumption. U is a smooth complex variety, contained in a smooth complete variety X, with complement Y a simple normal crossing divisor. Also assume that the irreducible components $Y_1, \cdots, Y_r$ of Y are totally ordered.

For a subset I of $\{1, \cdots, r\}$, set $Y_I = \cap_{i \in I} Y_i$ and $Y_\emptyset = X$. If $J \supset I$, there is an inclusion $Y_J \to Y_I$.

For $0 \leq a \leq r$ we set $Y^{(a)} = \coprod_{|I|=a} Y_I$ and $Y^{(0)} = X$. If $k = 0, \cdots, a$, the sum of the inclusions $Y_{i_1 \cdots i_a} \to Y_{i_1 \cdots \widehat{i_k}, \cdots i_a}$ gives a map $d_k : Y^{(a)} \to Y^{(a-1)}$.

(4.2) Recall from §3 that $\mathcal{C}_{X,\bullet}$ denotes the complex of sheaves of semi-analytic chains on X

$$0 \to \mathcal{C}_{2n} \to \mathcal{C}_{2n-1} \to \cdots \to \mathcal{C}_0 \to 0$$

which is concentrated in homological degree $[0, 2n]$. One has $\mathcal{C}_{X,p}(U) = C_p(U)$ for U open in X. We also use the notation $\mathcal{C}_\bullet(X)$ instead of $\mathcal{C}_{X,\bullet}$; but we avoid expressions such as $\mathcal{C}_\bullet(X)(U)$.

We view it as a cohomological complex concentrated in degree $[-2n, 0]$, and then shift it to define the complex $\mathcal{C}_X^\bullet$ as

$$\mathcal{C}_X^\bullet = \mathcal{C}_{X,\bullet}[-2n]\,.$$

It is concentrated in cohomological degree $[0, 2n]$; $\mathcal{C}_X^p$ consists of chains of real codimension p. The fundamental orientation chain of X gives an augmentation map $\mathbb{Q}_X \to \mathcal{C}_X^\bullet$, which is a quasi-isomorphism.

Instead of $\mathcal{C}_X^\bullet$ we may write $\mathcal{C}^\bullet(X)$, or just $\mathcal{C}(X)$ dropping the superscript; on the contrary we will not drop the subscript from $\mathcal{C}_\bullet(X)$.

Suppose X is complete and smooth. If $i : Y \subset X$ is the inclusion of a smooth divisor one has a map of sheaves complexes on X, namely $i_* : \mathcal{C}^\bullet(Y)[-2] \to \mathcal{C}^\bullet(X)$. As usual we have identified a sheaf $\mathcal{F}$ on Y as a sheaf $i_*\mathcal{F}$ on X.

Next assume $Y = Y_1 + \cdots + Y_r$ is a normal crossing divisor on X. For $J \supset I$ with $|J| = |I| + 1$, one has the map $i_* : \mathcal{C}^\bullet(Y_J)[-2] \to \mathcal{C}^\bullet(Y_I)$. If $I = \{i_1, \cdots, i_a\}$ with $i_1 < \cdots < i_a$ and $J = \{i_1, \cdots, i_k, j, i_{k+1}, \cdots, i_a\}$ also in the increasing order, set $\epsilon(J, I) = (-1)^k$. Then define the map $\delta : \mathcal{C}(Y^{(a+1)})[-2] \to \mathcal{C}(Y^{(a)})$ as follows: for a section $T = (T_J)$ of $\mathcal{C}^{\bullet-2}(Y^{(a+1)})$,

$$(\delta T)_I := \sum_{J \supset I} \epsilon(J, I)\, i_*(T_J) \in \mathcal{C}^\bullet(Y^{(a)})\,.$$

One verifies the identity $\delta\delta = 0$ and obtains a complex of complexes

$$0 \to \mathcal{C}(Y^{(r)})[-2r] \xrightarrow{\delta} \mathcal{C}(Y^{(r-1)})[-2r+2] \to \cdots \to \mathcal{C}(Y^{(1)})[-2] \xrightarrow{\delta} \mathcal{C}(X) \to 0$$

($\mathcal{C}(X)$ placed in degree 0). The total complex of which is denoted by $\mathcal{C}^\bullet(X \backslash Y)$, the differential of which is the sum of δ and $(-1)^a d$ on $\mathcal{C}(Y^{(a)})$. This is just a notation; note it is different from $\mathcal{C}^\bullet_{X\backslash Y}$ which a complex of sheaves on $X \backslash Y$.

We equip it with an increasing filtration W given by

$$W_m \mathcal{C}(X \backslash Y) = \big[0 \to \cdots \to 0 \to \mathcal{C}(Y^{(m)})[-2m] \to \cdots \to \mathcal{C}(X)\big]\,.$$

for $0 \le m \le r$ and by $W_{-1} = 0$, $W_m = \mathcal{C}(X \backslash Y)$ for $m \ge r$.

(4.3) For a c-soft sheaf $\mathcal{F}$ of vector spaces over a field Λ, let $D(\mathcal{F})$ be its dual sheaf, which by definition is given by the assignment $U \mapsto \Gamma_c(U, \mathcal{F})^*$, the linear dual of the Λ-vector space $\Gamma_c(U, \mathcal{F})$. One may apply the functor D to a complex of c-soft sheaves. We also define $\mathcal{D}(\mathcal{F}) = D(\mathcal{F})[-2n]$.

Let $\mathcal{A}^\bullet_X$ be the complex of sheaves of C^∞-forms on X. We define a complex of sheaves $\mathcal{D}_X$ by

$$\mathcal{D}_X = \mathcal{D}(\mathcal{A}_X) = D(\mathcal{A}_X)[-2n]$$

and call it the complex of *(algebraic) currents*. Note $\mathcal{D}^i_X = D(\mathcal{A}_X^{-i+2n})$.

The complex $\mathcal{D}^\bullet_X$ is concentrated in cohomological degree $[0, 2n]$, and there is a canonical map $\Phi : \mathcal{C}^\bullet_X \to \mathcal{D}^\bullet_X$, that sends α to the current

$$\int_\alpha : \quad \omega \mapsto \int_\alpha \omega\,.$$

It is a quasi-isomorphism. We also write $\mathcal{D}^\bullet(X)$ or $\mathcal{D}(X)$ for $\mathcal{D}_X$.

For a smooth divisor $Y \subset X$ there is a canonical map of complexes $\mathcal{D}(Y)[-2] \to \mathcal{D}(X)$. Thus for $Y = Y_1 + \cdots + Y_r$ a normal crossing divisor, there is a complex of complexes

$$\mathcal{D}(Y^{(r)})[-2r] \to \mathcal{D}(Y^{(r-1)})[-2r+2] \to \cdots \to \mathcal{D}(Y^{(1)})[-2] \to \mathcal{D}(X)\,,$$

the total complex will be denoted $\mathcal{D}(X \backslash Y)$. On this complex is there a filtration W given by

$$W_m \mathcal{D}(X \backslash Y) = \big[0 \to \cdots \to 0 \to \mathcal{D}(Y^{(m)})[-2m] \to \cdots \to \mathcal{D}(X)\big]\,.$$

Denote by $\tau = \tau_{\leq \bullet}$ the canonical filtration on any complex in an abelian category (see [De], §1.4). It is an increasing filtration, functorial for maps of complexes.

(4.4) **Proposition.** *One has inclusion* $(\mathcal{C}(X\backslash Y), \tau) \hookrightarrow (\mathcal{C}(X\backslash Y), W)$, *which is a filtered quasi-isomorphism. Similarly one has a filtered quasi-isomorphism* $(\mathcal{D}(X\backslash Y), \tau) \hookrightarrow (\mathcal{D}(X\backslash Y), W)$.

Proof. The complex in question is of the form

$$0 \to K_r \to \cdots \to K_a \to \cdots \to K_0 \to 0$$

with K_a placed in degree $-a$, and if the total complex K is equipped with filtration $W_\bullet$ given by

$$W_m K = [\cdots \to 0 \to K_m \to \cdots \to K_0 \to 0] \ .$$

The a-th column $K_a = \mathcal{C}(Y^{(a)})[-2a]$ is concentrated in degree $\geq 2a$. One has $K^i = \bigoplus_a K_a^{i+a}$. If the intersection $(\tau_{\leq m} K)^i \cap K_a^{i+a}$ is nonzero, one must have $m \geq i$ and $i + a \geq 2a$, hence $m \geq a$. This shows the inclusion $\tau_{\leq m} K \subset W_m K$.

The associated spectral sequence is of the form

$$E_1^{p,q} = \underline{H}^q(K_{-p}) \Rightarrow \underline{H}^{p+q}(K)\,.$$

For the complex $\mathcal{C}(X\backslash Y)$, we have

$$E_1^{p,q} = \underline{H}^{q+2p}(\mathcal{C}(Y^{(-p)})) \Rightarrow \underline{H}^{p+q}(\mathcal{C}(X\backslash Y))$$

where

$$E_1^{p,q} = \begin{cases} \mathbb{Q}_{Y^{(-p)}} & \text{if} \quad q = -2p, \quad -r \leq p \leq 0 \\ 0 & \text{if} \quad q \neq -2p\,. \end{cases}$$

It degenerates at E_1 and gives

$$\underline{H}^i(\mathcal{C}(X\backslash Y)) = E_1^{-i,2i} = \begin{cases} \mathbb{Q}_{Y^{(i)}} & \text{if} \quad 0 \leq i \leq r \\ 0 & \text{otherwise} \end{cases}$$

For the subcomplex $W_m\mathcal{C}(X\backslash Y)$ there is a similar spectral sequence, with terms

$$E_1^{p,q} = \begin{cases} \mathbb{Q}_{Y^{(-p)}} & \text{if} \quad q = -2p, \quad -m \leq p \leq 0 \\ 0 & \text{if} \quad q \neq -2p \end{cases}$$

converging to the cohomology of $W_m\mathcal{C}(X\backslash Y)$. It follows

$$\underline{H}^i(W_m\mathcal{C}(X\backslash Y)) = \begin{cases} \mathbb{Q}_{Y^{(i)}} & \text{if} \quad 0 \leq i \leq m \\ 0 & \text{otherwise} \end{cases}$$

Since this coincides with $\underline{H}^i(\tau_{\leq m}\mathcal{C}(X\backslash Y))$, the inclusion $\tau_{\leq m}\mathcal{C}(X\backslash Y) \to W_m\mathcal{C}(X\backslash Y)$ is a quasi-isomorphism.

The proof is the same for the complex $\mathcal{D}(X\backslash Y)$. $\square$

(4.5) For this subsection details may be found in [De], II, §3. Let $\Omega_X^\bullet\langle Y\rangle$ be the logarithmic de Rham complex of (X, Y), namely the sheaf of holomorphic forms on X with logarithmic singularities along Y. There is an increasing filtration $W_\bullet$ on this complex, called the weight filtration. There exists an isomorphism of complexes

$$\mathrm{Gr}_m^W \Omega_X^\bullet\langle Y\rangle \to \Omega_{Y^{(m)}}^\bullet[-m]$$

given by Poincaré residue. Since we have assumed the irreducible components of Y are totally ordered, there is no ambiguity of signs. One also has the Hodge filtration $F^\bullet$ on the complex $\Omega_X^\bullet\langle Y\rangle$.

Let $j : U = X - Y \to X$ be the inclusion. One has inclusions of filtered complexes

$$(\Omega_X^\bullet\langle Y\rangle, W) \leftarrow (\Omega_X^\bullet\langle Y\rangle, \tau) \to (j_*\Omega_U^\bullet, \tau)\,. \tag{4.5.1}$$

A fact of fundamental importance ([De], (3.18)) states that both maps are filtered quasi-isomorphisms.

Thanks to the Malgrange preparation theorem we also have the following (cf. [De], II, §3.2). Let $\mathcal{A}_X\langle Y\rangle$ be its $\bar\partial$-resolution, namely the total complex of the double complex $\Omega_X^\bullet\langle Y\rangle \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}$. Thus it is the sheaf of smooth forms with logarithmic poles along Y. We also write $\mathcal{A}(X)\langle Y\rangle$ for $\mathcal{A}_X\langle Y\rangle$.

The filtration W on $\Omega_X^\bullet\langle Y\rangle$ induces a filtration W on the complex $\mathcal{A}_X\langle Y\rangle$; we have $\mathrm{Gr}^W\mathcal{A}_X\langle Y\rangle \cong \mathrm{Gr}^W\Omega_X^\bullet\langle Y\rangle \otimes_{\mathcal{O}_X} \Omega_X^{0,\bullet}$, in particular it is a complex of fine sheaves.

Similarly one has the induced Hodge filtration F on $\mathcal{A}_X\langle Y\rangle$. The objects $\mathrm{Gr}_F\mathcal{A}_X\langle Y\rangle$ and $\mathrm{Gr}_F\mathrm{Gr}^W\mathcal{A}_X\langle Y\rangle$ are complexes of fine sheaves.

(4.6) One has a map of complexes $\mathcal{A}(X) \to \mathcal{D}(X)$ which sends a smooth form to the corresponding current. It extends to a map $\mathcal{P} : \mathcal{A}(X)\langle Y\rangle \to \mathcal{D}(X)$ which sends a form ω to the current $[\omega]$ it represents:

$$[\omega](\psi) = \int_X \omega \wedge \psi\,,$$

the integral being convergent since ω is locally integrable. This $\mathcal{P}$ is not a map of complexes; the failure is explained by the residue formula given below.

For each Y_I of Y, one has a map of complexes $R_{Y_I} : \mathcal{A}(X)\langle Y\rangle \to \mathcal{A}(Y_I)\langle\widehat{Y_I}\rangle[-|I|]$ which sends a local section ω to its residue along Y_I. Here $\widehat{Y_I}$ is the divisor on Y_I given by

$$\sum Y_J \qquad (J \text{ varies over } J \supset I \text{ with } |J| = |I| + 1)\,.$$

In particular for each component Y_i of Y, one has residue $R_{Y_i} : \mathcal{A}(X)\langle Y\rangle \to \mathcal{A}(Y_i)\langle\widehat{Y_i}\rangle[-1]$.

We state the residue formula, which is easily verified: If ω is a section of $\mathcal{A}(X)\langle Y\rangle$ then one has

$$d[\omega] - [d\omega] = \sum [R_{Y_i}(\omega)]\,.$$

(4.7) Let $\gamma : \mathcal{A}(X)\langle Y\rangle \to \mathcal{D}(X\backslash Y)$ be a map given as follows. For each Y_I with $a = |I|$ we have a sequence of maps

$$\mathcal{A}(X)\langle Y\rangle \xrightarrow{R_{Y_I}} \mathcal{A}(Y_I)\langle \widehat{Y_I}\rangle \to \mathcal{D}(Y_I) \subset \mathcal{D}(Y^{(a)})$$

which sends a local section ω of $\mathcal{A}(X)\langle Y\rangle$ to $[R_{Y_I}(\omega)] \in \mathcal{D}(Y_I)$. By definition

$$\gamma(\omega) = \sum_I (-1)^{|I|} [R_{Y_I}(\omega)] \in \mathcal{D}(X\backslash Y) \tag{4.7.1}$$

the sum over all indices I.

Proposition. *The above construction gives a map of complexes $\gamma : \mathcal{A}(X)\langle Y\rangle \to \mathcal{D}(X\backslash Y)$. Further it is a filtered quasi-isomorphism with respect to the filtrations W.*

Proof. If $R_{I,J} : \mathcal{A}(Y_I)\langle \widehat{Y_I}\rangle \to \mathcal{A}(Y_J)\langle \widehat{Y_J}\rangle$ is the residue map then with a and k as before, one verifies

$$R_{I,J} R_I = (-1)^{a+k} R_J : \mathcal{A}(X)\langle Y\rangle \to \mathcal{A}(Y_J)\langle \widehat{Y_J}\rangle\,.$$

Also one has $R_{Y_I} d = (-1)^a d R_{Y_I}$. Hence the assertion in question is reduced to the residue formula:

$$d[R_{Y_I}(\omega)] - [dR_{Y_I}(\omega)] = \sum_{J\supset I} [R_{I,J} R_{Y_I}(\omega)]\,.$$

$\square$

(4.8) Let $j = U = X\backslash Y \to X$ be the inclusion. The map $\mathcal{C}(X) \to j_*\mathcal{C}_U$ given by $\alpha \mapsto \alpha|U$, see (3.10), induces a map of complexes

$$\mathcal{C}(X\backslash Y) \to j_*\mathcal{C}_U$$

which is a quasi-isomorphism. On the complex $\mathcal{C}(X\backslash Y)$ there is an inclusion of filtrations

$$(\mathcal{C}(X\backslash Y), \tau) \hookrightarrow (\mathcal{C}(X\backslash Y), W)$$

which is a filtered quasi-isomorphism.

Similarly one has a map $\mathcal{D}(X\backslash Y) \to j_*\mathcal{D}_U$ and the inclusion

$$(\mathcal{D}(X\backslash Y), \tau) \hookrightarrow (\mathcal{D}(X\backslash Y), W)$$

which is a filtered quasi-isomorphism.

We have thus a commutative square of complexes on X

$$\begin{array}{ccc}
(j_*\mathcal{C},\tau) & \longrightarrow & (j_*\mathcal{D},\tau) \\
\uparrow & & \uparrow \\
(\mathcal{C}(X\backslash Y),\tau) & \longrightarrow & (\mathcal{D}(X\backslash Y),\tau) \\
\downarrow & & \downarrow \\
(\mathcal{C}(X\backslash Y),W) & \longrightarrow & (\mathcal{D}(X\backslash Y),W)
\end{array}$$

with arrows filtered quasi-isomorphisms.

(4.9) Let $\mathcal{S}^\bullet$ be the differential sheaf of smooth singular cochains (see [Br], p.26); in this paper we always take the coefficient ring to be $\mathbb{Q}$. We know that $\mathcal{S}^\bullet$ is a complex of flabby sheaves and the augmentation map $\mathbb{Q} \to \mathcal{S}^\bullet$ is a resolution, see [Ha].

Let $D(\mathcal{S}^\bullet)$ be the dual of $\mathcal{C}^\bullet$. The canonical map (given by integration on chains) $c : \mathcal{A}^\bullet \to \mathcal{S}^\bullet$ induces a map $\chi : D(\mathcal{S}^\bullet) \to D(\mathcal{A}^\bullet)$.

There is the cap product pairing

$$\cap : D(\mathcal{S}^\bullet) \otimes \mathcal{S}^\bullet \to D(\mathcal{S}^\bullet)$$

induced from the cup product on $\mathcal{S}^\bullet$, as in [Br], V-(10.3).

If $f_X \in \Gamma(X, D(\mathcal{S}^{2n}))$ is a cocycle representing the fundamental class of X, one has a map of complexes

$$\kappa := f_X \cap (-) : \mathcal{S}^\bullet \to D(\mathcal{S}^\bullet)[-2n]\,.$$

If we give X a triangulation, then the orientation cycle $[X] = \sum \sigma$ provides such a cochain.

The diagram

$$\begin{array}{ccc}
\mathcal{S}^\bullet & \xleftarrow{c} & \mathcal{A}^\bullet \\
{\scriptstyle\kappa}\downarrow & & \downarrow \\
D(\mathcal{S}^\bullet)[-2n] & \xrightarrow{\chi} & \mathcal{D}^\bullet
\end{array}$$

(recall the map on the right sends a form ψ to the current $[\psi]$) is homotopy commutative.

(4.10) One has a map

$$\xi : \mathcal{C}_\bullet \to D(\mathcal{S}^\bullet)$$

given follows. In view of the convention (4.2) this really means a map For $\alpha \in \mathcal{C}_p(U)$ and $u \in \mathcal{S}^p(U)$, with $\mathrm{supp}(u) = K$, write $\alpha = \alpha' + \alpha''$ where α' is compactly supported, and $\mathrm{supp}(\alpha'') \subset U - K$; define $\xi(\alpha) \in D(\mathcal{S}^p)(U)$ by

$$\langle \xi(\alpha), u\rangle = \langle \alpha', u\rangle\,.$$

The map ξ is a quasi-isomorphism.

Recall the map $\Phi : \mathcal{C}_\bullet \to \mathcal{D}^\bullet[2n]$ given by integration. One verifies that the diagram

$$\begin{array}{ccc} D(\mathcal{S}^\bullet) & \xrightarrow{\chi} & \mathcal{D}^\bullet[2n] \\ {\scriptstyle\xi}\big\uparrow & & \| \\ \mathcal{C}_\bullet & \xrightarrow{\Phi} & \mathcal{D}^\bullet[2n] \end{array}$$

commutes.

5 The Hodge complexes and comparison

(5.1) We refer to [Br], II-§9 for the basic notion of family of supports Φ, and the class of Φ-soft, Φ-fine and Φ-acyclic sheaves. Suppose Φ is a paracompactifying family of supports on a space X. Any flabby sheaf is Φ-soft, any Φ-fine sheaf is Φ-soft, and any Φ-soft sheaf is Φ-acyclic.

If $\mathcal{A}$ is Φ-soft and torsion-free over a base ring L, then for any sheaf $\mathcal{B}$ over L, $\mathcal{A} \otimes_L \mathcal{B}$ is Φ-soft.

(5.2) **Definition.** With the definitions and notions in the previous sections, consider the triple of filtered complexes of sheaves on X

$$(\mathcal{C}(X\backslash Y), W) \longrightarrow (\mathcal{D}(X\backslash Y), W) \xleftarrow{\ \gamma\ } (\mathcal{A}_X\langle Y\rangle, (W, F)\,) \tag{5.2.1}$$

and take the global section on X; we get a triple of filtered complexes of vector spaces

$$\Gamma(X, (\mathcal{C}(X\backslash Y), W)\,) \to \Gamma(X, (\mathcal{D}(X\backslash Y), W)\,) \leftarrow \Gamma(X, (\mathcal{A}_X\langle Y\rangle, (W, F))\,)\,. \tag{5.2.2}$$

The third term has the Hodge filtration $F^\bullet$ as an additional filtration.

The triple (5.2.2) is a $\mathbb{Q}$-mixed Hodge complex, and hence (5.2.1) is a $\mathbb{Q}$-mixed Hodge complex of sheaves. By definition *either* of these is called the *homological mixed Hodge complex* of $X\backslash Y$, and denoted by $\mathbb{L}(X\backslash Y)$.

(5.3) We recall the "standard" construction of the Hodge complex for $X\backslash Y$. Let $j : U = X\backslash Y \to X$ be the inclusion of the open set.

To a complex of sheaves $\mathcal{F}^\bullet$ on U, one may apply the functor of Godement resolution on U, $\mathsf{C}^\bullet(-) = \mathsf{C}^\bullet_U(-)$, and then apply the functor j_* to get a complex of sheaves $j_*\mathsf{C}^\bullet(\mathcal{F}^\bullet)$ on X. Note that $\mathbb{R}j_*\mathcal{F}^\bullet = j_*\mathsf{C}^\bullet(\mathcal{F}^\bullet)$ in the derived category.

The canonical maps $\mathcal{A}^\bullet_X\langle Y\rangle \to j_*\mathcal{A}^\bullet \to j_*\mathsf{C}^\bullet(\mathcal{A})$ are quasi-isomorphisms. There results a triple of complexes of sheaves on X and quasi-isomorphisms

(1) $$j_*\mathsf{C}^\bullet(\mathbb{Q}) \longrightarrow j_*\mathsf{C}^\bullet(\mathcal{A}) \longleftarrow \mathcal{A}^\bullet_X\langle Y\rangle\,.$$

If we equip each complex with the canonical filtration $\tau_\leq$, the two maps are filtered quasi-isomorphisms. We take the latter filtered quasi-isomorphism $(\mathcal{A}^\bullet_X\langle Y\rangle, \tau) \to (j_*\mathsf{C}^\bullet(\mathcal{A}), \tau)$, and take its push-out along the filtered quasi-isomorphism $(\mathcal{A}^\bullet_X\langle Y\rangle, \tau) \to (\mathcal{A}^\bullet_X\langle Y\rangle, W)$ (see [De]):

$$\begin{array}{ccc} (\mathcal{A}^\bullet_X\langle Y\rangle, \tau) & \longrightarrow & (j_*\mathsf{C}^\bullet(\mathcal{A}), \tau) \\ \downarrow & & \downarrow \\ (\mathcal{A}^\bullet_X\langle Y\rangle, W) & \longrightarrow & (j_*\mathsf{C}^\bullet(\mathcal{A}), \tau)^\triangle\,. \end{array}$$

The four maps are filtered quasi-isomorphisms. We obtain a diagram

$$\begin{array}{lccccc} (1) & (j_*\mathsf{C}^\bullet(\mathbb{Q}), \tau) & \longrightarrow & (j_*\mathsf{C}^\bullet(\mathcal{A}), \tau) & \longleftarrow & (\mathcal{A}^\bullet_X\langle Y\rangle, \tau) \\ & \| & & \downarrow & & \downarrow \\ (1)^\triangle & (j_*\mathsf{C}^\bullet(\mathbb{Q}), \tau) & \longrightarrow & (j_*\mathsf{C}^\bullet(\mathcal{A}), \tau)^\triangle & \longleftarrow & (\mathcal{A}^\bullet_X\langle Y\rangle, W)\,. \end{array}$$

Take the second row of the diagram; apply the functor $\mathsf{C}^\bullet = \mathsf{C}^\bullet_X$ of Godement resolution over X to the first and second terms; the third term $\mathcal{A}^\bullet_X\langle Y\rangle$ has additionally the Hodge filtration $F^\bullet$. Thus we obtain a triple of filtered complexes

$$(1)^\triangle \qquad \mathsf{C}^\bullet(j_*\mathsf{C}^\bullet(\mathbb{Q}),\tau) \longrightarrow \mathsf{C}^\bullet((j_*\mathsf{C}^\bullet(\mathcal{A}),\tau)^\triangle) \longleftarrow (\mathcal{A}^\bullet_X\langle Y\rangle, W, F)\,. \qquad (5.3.1)$$

This is a $\mathbb{Q}$-Hodge complex of sheaves, namely if we apply the global section functor $\Gamma(X,-)$, we get a $\mathbb{Q}$-Hodge complex

$$\Gamma(X,\mathsf{C}^\bullet(j_*\mathsf{C}^\bullet(\mathbb{Q}),\tau)) \to \Gamma\left(X,\mathsf{C}^\bullet((j_*\mathsf{C}^\bullet(\mathcal{A}),\tau)^\triangle)\right) \leftarrow \Gamma(X,(\mathcal{A}^\bullet_X\langle Y\rangle, W, F\,))\,. \qquad (5.3.2)$$

This is the $\mathbb{Q}$-mixed Hodge complex due essentially to Deligne and Beilinson, that gives mixed Hodge structure on the cohomology of $X\backslash Y$; We call (5.3.2), (resp. (5.3.1)), the *standard* mixed Hodge complex (resp. mixed Hodge complex of sheaves) for $X\backslash Y$. We employ the notation $\mathbb{K}(X\backslash Y)$ for either of them.

The proof of the following theorem is the aim of this section.

(5.4) **Theorem.** *There is a canonical isomorphism between the two mixed Hodge complexes of sheaves $\mathbb{K}(X\backslash Y)$ and $\mathbb{L}(X\backslash Y)$.*

(5.5) The augmentation map $\mathbb{C}_X \to \mathcal{A}^\bullet$ is a quasi-isomorphism. Since $\mathcal{A}^\bullet$ is a complex of $\mathbb{Q}$-vector spaces (hence flat) and bounded, $\mathcal{S}^\bullet \otimes (-)$ gives a quasi-isomorphism $\alpha : \mathcal{S}^\bullet \to \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet$, [Hart], II, Lemma 4.1 with conditions a) and 2).

Similarly the augmentation map $\mathbb{Q}_X \to \mathcal{S}^\bullet$ is a quasi-isomorphism. Since $\mathcal{A}^\bullet$ is a complex of bounded $\mathbb{Q}$-vector spaces, $\mathcal{A}^\bullet \otimes (-)$ gives a quasi-isomorphism $\beta : \mathcal{S}^\bullet \to \mathcal{S}^\bullet \otimes_{\mathbb{Q}} \mathcal{A}^\bullet$, the same lemma with conditions b) and 2).

The complexes $\mathcal{S}^\bullet$ and $\mathcal{A}^\bullet$ are $\Gamma(X,-)$-acyclic as well as f_*-acyclic for any map $f : X \to X'$.

Consider these complexes on $U = X\backslash Y$. We have a commutative square

$$\begin{tikzcd}
\mathbb{Q} \arrow[r] \arrow[d] & \mathcal{A}^\bullet \arrow[d, "\beta"] \\
\mathcal{S}^\bullet \arrow[r, "\alpha"] & \mathcal{S}^\bullet \otimes \mathcal{A}
\end{tikzcd}$$

This gives us a commutative diagram of complexes of sheaves on U

$$\begin{tikzcd}
\mathsf{C}^\bullet(\mathbb{Q}) \arrow[r] \arrow[d] & \mathsf{C}^\bullet(\mathcal{A}) \arrow[r] \arrow[d] & \mathcal{A}^\bullet \arrow[d, "id"] \\
\mathsf{C}^\bullet(\mathcal{S}^\bullet) \arrow[r] & \mathsf{C}^\bullet(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[r] & \mathcal{A}^\bullet \\
\mathcal{S}^\bullet \arrow[r] \arrow[u] & \mathcal{S}^\bullet \otimes \mathcal{A}^\bullet \arrow[r] \arrow[u] & \mathcal{A}^\bullet\,. \arrow[u, "id"']
\end{tikzcd}$$

If we apply the functor j_*, and replace $j_*\mathcal{A}^\bullet$ with $\mathcal{A}_X\langle Y\rangle$ via the quasi-isomorphism $\mathcal{A}_X\langle Y\rangle \to j_*\mathcal{A}^\bullet$, we obtain a commutative diagram on X

$$\begin{tikzcd}
(1) & j_*\mathsf{C}^\bullet(\mathbb{Q}) \arrow[r] \arrow[d] & j_*\mathsf{C}^\bullet(\mathcal{A}) \arrow[d] & \mathcal{A}^\bullet_X\langle Y\rangle \arrow[l] \arrow[d, equal] \\
(2) & j_*\mathsf{C}^\bullet(\mathcal{S}) \arrow[r] & j_*\mathsf{C}^\bullet(\mathcal{S}\otimes\mathcal{A}) & \mathcal{A}^\bullet_X\langle Y\rangle \arrow[l] \arrow[d, equal] \\
(3) & j_*(\mathcal{S}) \arrow[r, "\alpha"] \arrow[u] & j_*(\mathcal{S}\otimes\mathcal{A}) \arrow[u] & \mathcal{A}^\bullet_X\langle Y\rangle\,. \arrow[l, "\beta"']
\end{tikzcd}$$

As we produced $(1)^\triangle$ out of (1) by means of push-out along $(\mathcal{A}_X\langle Y\rangle,\tau) \to (\mathcal{A}_X\langle Y\rangle, W)$, by the same process we get triples out of row (2) and row (3); by functoriality of push-out (2.1.2) we obtain a commutative diagram of filtered complexes

$$\begin{tikzcd}
(1)^\triangle & (j_*\mathsf{C}^\bullet(\mathbb{Q}),\tau) \arrow[r] \arrow[d] & (j_*\mathsf{C}^\bullet(\mathcal{A}),\tau)^\triangle \arrow[d] & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l] \arrow[d, equal] \\
(2)^\triangle & (j_*\mathsf{C}^\bullet(\mathcal{S}),\tau) \arrow[r] & (j_*\mathsf{C}^\bullet(\mathcal{S}\otimes\mathcal{A}),\tau)^\triangle & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l] \arrow[d, equal] \\
(3)^\triangle & (j_*(\mathcal{S}),\tau) \arrow[r, "\alpha"] \arrow[u] & (j_*(\mathcal{S}\otimes\mathcal{A}),\tau)^\triangle \arrow[u] & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l, "\beta'"']
\end{tikzcd}$$

where the maps $(1)^\triangle \to (2)^\triangle \leftarrow (3)^\triangle$ are quasi-isomorphisms of triples of filtered complexes.

Further, taking Godement resolution for the first and second terms, and equipping the third term with the Hodge filtration $F^\bullet$, we have

$$\begin{tikzcd}
(1)^\triangle & \mathsf{C}^\bullet(j_*\mathsf{C}^\bullet(\mathbb{Q}),\tau) \arrow[r] \arrow[d] & \mathsf{C}^\bullet((j_*\mathsf{C}^\bullet(\mathcal{A}),\tau)^\triangle\,) \arrow[d] & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l] \arrow[d, equal] \\
(2)^\triangle & \mathsf{C}^\bullet(j_*\mathsf{C}^\bullet(\mathcal{S}),\tau) \arrow[r] & \mathsf{C}^\bullet((j_*\mathsf{C}^\bullet(\mathcal{S}\otimes\mathcal{A}),\tau)^\triangle\,) & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l] \arrow[d, equal] \\
(3)^\triangle & \mathsf{C}^\bullet(j_*(\mathcal{S}),\tau) \arrow[r, "\alpha"] \arrow[u] & \mathsf{C}^\bullet((j_*(\mathcal{S}\otimes\mathcal{A}),\tau)^\triangle\,) \arrow[u] & (\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,) \arrow[l, "\beta'"']
\end{tikzcd}$$

Each row is a mixed Hodge complex of sheaves.

If we apply the functor Γ to $(2)^\triangle$, we obtain a Hodge complex of the form

$$\Gamma(X,\mathsf{C}^\bullet(j_*\mathcal{S}^\bullet,\tau)\,) \to \Gamma\left(X,\mathsf{C}^\bullet((j_*(\mathcal{S}^\bullet\otimes\mathcal{A}^\bullet),\tau)^\triangle)\right) \leftarrow \Gamma(X,\mathsf{C}^\bullet(\mathcal{A}^\bullet_X\langle Y\rangle,(W,F)\,)\,)\,.$$

The same holds for $(3)^\triangle$.

(5.6) Recall the maps $c:\mathcal{A}^\bullet\to\mathcal{S}^\bullet$, $\kappa:\mathcal{S}^\bullet\to\mathcal{D}(\mathcal{S}^\bullet)$, and $\chi:\mathcal{D}(\mathcal{S}^\bullet)\to\mathcal{D}^\bullet$. The map c is the obvious one which takes a form ψ to the singular cochain it defines.

We consider the map

$$\lambda = \chi\circ\kappa\circ(1\cup c) : \mathcal{S}^\bullet\otimes_{\mathbb{Q}}\mathcal{A}^\bullet \to \mathcal{D}^\bullet\,.$$

One has a diagram of maps of complexes:

$$\begin{tikzcd}
(3) & \mathcal{S}^\bullet \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & \mathcal{S}^\bullet\otimes\mathcal{A}^\bullet \arrow[d, "\lambda"'] & \mathcal{A}^\bullet \arrow[l, "\beta"'] \arrow[d, "id"] \\
(4) & \mathcal{D}(\mathcal{S}^\bullet) \arrow[r, "\chi"] & \mathcal{D}^\bullet & \mathcal{A}^\bullet\,. \arrow[l, "\mathcal{P}"']
\end{tikzcd} \tag{i}$$

The left square commutes, and the right square commutes up to homotopy, namely there is a map of degree -1, $S : \mathcal{A}^\bullet \to \mathcal{D}^{\bullet -1}$ such that

$$dS + Sd = -\lambda\beta + \mathcal{P}\,.$$

We also have a natural map $\xi : \mathcal{C}_\bullet \to D(\mathcal{S})$ and a commutative diagram

$$\begin{array}{lccccc}
(4) & \mathcal{D}(\mathcal{S}^\bullet) & \xrightarrow{\chi} & \mathcal{D}^\bullet & \xleftarrow{\mathcal{P}} & \mathcal{A}^\bullet \\
 & \xi\Big\uparrow & & id\Big\uparrow & & \Big\uparrow id \\
(5) & \mathcal{C}_\bullet & \xrightarrow{\Phi} & \mathcal{D}^\bullet & \xleftarrow{\mathcal{P}} & \mathcal{A}^\bullet\,.
\end{array} \qquad (ii)$$

In the preceding subsection we took the first row of diagram (i) and turned it to a triple of filtered complexes; now we apply the same process to the other rows of (i) and (ii).

First look at (i). Apply the functor j_* to get a diagram

$$\begin{tikzcd}
j_*\mathcal{S}^\bullet \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & j_*(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet) \arrow[d, "\lambda"'] & j_*\mathcal{A}^\bullet \arrow[l, "\beta"'] \arrow[d, "id"] \\
j_*\mathcal{D}(\mathcal{S}^\bullet) \arrow[r, "\chi"] & j_*\mathcal{D}^\bullet & j_*\mathcal{A}^\bullet\,. \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

Here $j_*\mathcal{S}^\bullet$, for example, is short for $j_*(\mathcal{S}^\bullet_U)$. The left square commutes; for the right square there is a map $S : j_*\mathcal{A}^\bullet \to j_*(\mathcal{D}^{\bullet-1})$ with identity $dS + Sd = -\lambda\beta + \mathcal{P}$.

Equip each of the complexes with the canonical filtration $\tau_\leq$; then we have a diagram of filtered complexes

$$\begin{tikzcd}
(j_*\mathcal{S}^\bullet, \tau) \arrow[r, "\alpha"] \arrow[d, "\kappa"'] & (j_*(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet), \tau) \arrow[d, "\lambda"'] & (j_*\mathcal{A}^\bullet, \tau) \arrow[l, "\beta"'] \arrow[d, "id"] \\
(j_*\mathcal{D}(\mathcal{S}^\bullet), \tau) \arrow[r, "\chi"] & (j_*\mathcal{D}^\bullet, \tau) & (j_*\mathcal{A}^\bullet, \tau)\,. \arrow[l, "\mathcal{P}"']
\end{tikzcd}$$

Further one verifies that the map S gives a filtered homotopy, namely it takes $\tau_{\leq m}(j_*\mathcal{A}^\bullet)$ into $\tau_{\leq m}(j_*(\mathcal{D}^{\bullet-1}))$.

Composing with the quasi-isomorphism $\mathcal{A}_X\langle Y\rangle \to j_*\mathcal{A}^\bullet$, both equipped with the canonical filtration, one obtains a diagram of filtered complexes

$$\begin{array}{lccccc}
(3) & (j_*\mathcal{S}^\bullet, \tau) & \xrightarrow{\alpha} & (j_*(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet), \tau) & \xleftarrow{\beta} & (\mathcal{A}_X\langle Y\rangle, \tau) \\
 & \kappa\Big\downarrow & & \lambda\Big\downarrow & & \Big\downarrow id \\
(4) & (j_*\mathcal{D}(\mathcal{S}^\bullet), \tau) & \xrightarrow{\chi} & (j_*\mathcal{D}^\bullet, \tau) & \xleftarrow{\mathcal{P}} & (\mathcal{A}_X\langle Y\rangle, \tau)\,.
\end{array}$$

with filtered homotopy S for the right hand square.

From (4) one obtains the push-out $(j_*\mathcal{D}^\bullet, \tau)^\triangle$, and in light of the functoriality of push-out (2.2) we have a *commutative* diagram

$$\begin{array}{lccccc}
(3)^\triangle & (j_*\mathcal{S}^\bullet, \tau) & \longrightarrow & (j_*(\mathcal{S}^\bullet \otimes \mathcal{A}^\bullet), \tau)^\triangle & \xleftarrow{\beta'} & (\mathcal{A}_X\langle Y\rangle, (W, F)\,) \\
 & \kappa\Big\downarrow & & \lambda^\triangle\Big\downarrow & & \Big\| \\
(4)^\triangle & (j_*\mathcal{D}(\mathcal{S}^\bullet), \tau) & \longrightarrow & (j_*\mathcal{D}^\bullet, \tau)^\triangle & \xleftarrow{\mathcal{P}'} & (\mathcal{A}_X\langle Y\rangle, (W, F)\,)\,.
\end{array}$$

Taking Godement resolutions for the first and second terms, we obtain mixed Hodge complexes of sheaves and a morphism between them:

$$\begin{array}{lccccc} (3)^{\triangle} & \mathsf{C}^\bullet((j_*\mathcal{S}^\bullet,\tau)) & \longrightarrow & \mathsf{C}^\bullet(j_*(\mathcal{S}^\bullet\otimes\mathcal{A}^\bullet),\tau)^{\triangle} & \xleftarrow{\beta'} & (\mathcal{A}_X\langle Y\rangle,(W,F)) \\ & \Big\downarrow{\scriptstyle\kappa} & & \Big\downarrow{\scriptstyle\lambda^{\triangle}} & & \| \\ (4)^{\triangle} & \mathsf{C}^\bullet(j_*\mathcal{D}(\mathcal{S}^\bullet),\tau) & \longrightarrow & \mathsf{C}^\bullet((j_*\mathcal{D}^\bullet,\tau)^{\triangle}) & \xleftarrow{\mathcal{P}'} & (\mathcal{A}_X\langle Y\rangle,(W,F)). \end{array}$$

Next we look at diagram (ii); we get a commutative diagram

$$\begin{array}{ccccc} j_*\mathcal{D}(\mathcal{S}^\bullet) & \xrightarrow{\chi} & j_*\mathcal{D}^\bullet & \xleftarrow{\mathcal{P}} & j_*\mathcal{A}^\bullet \\ \Big\uparrow{\scriptstyle\xi} & & \Big\uparrow{\scriptstyle id} & & \Big\uparrow{\scriptstyle id} \\ j_*\mathcal{C}^\bullet & \xrightarrow{\Phi} & j_*\mathcal{D}^\bullet & \xleftarrow{\mathcal{P}} & j_*\mathcal{A}^\bullet. \end{array}$$

We replace the complex $j_*\mathcal{A}^\bullet$ on the right with $\mathcal{A}_X\langle Y\rangle$; then with the canonical filtrations we have a commutative diagram of filtered complexes

$$\begin{array}{lccccc} (4) & (j_*\mathcal{D}(\mathcal{S}^\bullet),\tau) & \xrightarrow{\chi} & (j_*\mathcal{D}^\bullet,\tau) & \xleftarrow{\mathcal{P}} & (\mathcal{A}_X\langle Y\rangle,\tau) \\ & \Big\uparrow & & \Big\uparrow{\scriptstyle id} & & \Big\uparrow{\scriptstyle id} \\ (5) & (j_*\mathcal{C}^\bullet,\tau) & \xrightarrow{\Phi} & (j_*\mathcal{D}^\bullet,\tau) & \xleftarrow{\mathcal{P}} & (\mathcal{A}_X\langle Y\rangle,\tau). \end{array}$$

By taking push-out and then taking Godement resolutions we have obviously a commutative diagram

$$\begin{array}{lccccc} (4)^{\triangle} & \mathsf{C}^\bullet(j_*\mathcal{D}(\mathcal{S}^\bullet),\tau) & \longrightarrow & \mathsf{C}^\bullet((j_*\mathcal{D}^\bullet,\tau)^{\triangle}) & \xleftarrow{\mathcal{P}'} & (\mathcal{A}_X\langle Y\rangle,(W,F)) \\ & \Big\uparrow & & \Big\uparrow{\scriptstyle id} & & \| \\ (5)^{\triangle} & \mathsf{C}^\bullet(j_*\mathcal{C},\tau) & \longrightarrow & \mathsf{C}^\bullet(j_*\mathcal{D}^\bullet,\tau)^{\triangle}) & \xleftarrow{\mathcal{P}'} & (\mathcal{A}_X\langle Y\rangle,(W,F)). \end{array}$$

which is a morphism of mixed Hodge complex of sheaves. These give quasi-isomorphisms of mixed Hodge complexes $(3)^{\triangle}\to(4)^{\triangle}\leftarrow(5)^{\triangle}$.

(5.7) We also have a commutative diagram of filtered complexes

$$\begin{array}{lccccc} (5) & (j_*\mathcal{C},\tau) & \xrightarrow{\Phi} & (j_*\mathcal{D},\tau) & \xleftarrow{\mathcal{P}} & (\mathcal{A}^\bullet_X\langle Y\rangle,\tau) \\ & \Big\uparrow & & \Big\uparrow & & \| \\ (6) & (\mathcal{C}(X\backslash Y),\tau) & \xrightarrow{\Phi} & (\mathcal{D}(X\backslash Y),\tau) & \xleftarrow{\gamma} & (\mathcal{A}^\bullet_X\langle Y\rangle,\tau) \\ & \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ (7) & (\mathcal{C}(X\backslash Y),\tau) & \xrightarrow{\Phi} & (\mathcal{D}(X\backslash Y),W) & \xleftarrow{\gamma} & (\mathcal{A}^\bullet_X\langle Y\rangle,W) \end{array}$$

(the commutativity of the upper right corner is obvious). To the first two rows one may apply the push-out as before, and one has an induced map $(6)^{\triangle}\to(5)^{\triangle}$ (obvious functoriality of push-out). By the universal property (2.1.1), there is a map $(6)^{\triangle}\to(7)$, therefore we have filtered quasi-isomorphisms $(5)^{\triangle}\leftarrow(6)^{\triangle}\to(7)$.

This completes the proof of Theorem (5.4).

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