Blow-ups and mixed motives

Masaki Hanamura

Abstract

The blow-up formula for Chow groups of smooth varieties is known; for smooth projective varieties there is a similar formula for motives. We generalize these and prove blow-up formulas for higher Chow groups and for mixed motives of smooth quasi-projective varieties.

Introduction. In the theory of ordinary Chow groups, there are the projective bundle formula, the self-intersection formula, and the blow-up formula. The first proof of the self-intersection formula and the blow-up formula for the integral Chow group is due to A.T. Lascu, D. Mumford and D.B. Scott [LMS]; the argument is reproduced in [SGA]. A key idea is the use of the deformation to the normal cone. For a modern treatment see [Fu, Chap. 6]; the self-intersection and the blow-up formulas are part of the properties of the refined Gysin maps.

In §1 we consider these formulas for higher Chow groups $\mathrm{CH}^r(X,n)$. The statements are parallel to those for ordinary Chow groups (the case $n=0$). The projective bundle formula is known to hold. In (1.1) and (1.2), we consider the self-intersection formula and the blow-up formula; for the proof we almost follow that in [SGA], using now the localization sequence for higher Chow groups. From the blow-up formula one can derive the contravariant descent property for higher Chow groups, Theorem (1.5). This is used in [Ha 2] where we show the contravariant descent property for cubical hyperresolutions of a variety.

In the rest of this paper, where §1 is not be used except for the case $n=0$, we will:

(A) Formulate and prove the analogues of (1.1) and (1.2) for *relative motives*, and derive the original (1.1) and (1.2) from them;

(B) Derive from (A) the analogues of (1.1) and (1.2) for *mixed motives.*

In §2, we briefly recall the definition of $\mathcal{D}(k)$, the triangulated category of mixed motives over a field k, and, assuming the characteristic of k is zero, the construction of the functor h from the category of smooth quasi-projective varieties to $\mathcal{D}(k)$. To each smooth variety X there corresponds an object $h(X)$ (also denoted $L(X)$) of $\mathcal{D}(k)$, and to each map $f: X \to Y$ there is a corresponding morphism $f^*: h(Y) \to h(X)$.

Further, we show $X \mapsto h(X)$ is a "functor" on an appropriate correspondence category of smooth quasi-projective varieties: For a cycle u on $X \times Y$, which is proper over Y, there is an induced morphism $L(u): h(X)(r)[2r] \to h(Y)(s)[2s]$, if $\operatorname{codim} u = \dim X + s - r$. Here (r) denotes the Tate twist, and $[2r]$ is the shift in the triangulated category.

For example if $\alpha \in \mathrm{CH}^r(X)$ there corresponds a morphism $C(\alpha): h(X) \to h(X)(r)[2r]$. A proper map $f: X \to Y$ induces a morphism $f_*: h(X)(\dim X)[2\dim X] \to h(Y)(\dim Y)[2\dim Y]$. There is partial functoriality for morphisms $L(u)$ in the following sense. If Z is another smooth variety, v a cycle on $Y \times Z$ proper over Z, and if the composition $v \circ u$ is defined as a cycle on $X \times Z$, then one has $L(v \circ u) = L(v)L(u)$.

[0]2000 *Mathematics Subject Classification.* Primary 14C25; Secondary 14C15, 14C35.

We record some formulas involving f^*, f_* and $C(\alpha)$, recall the localization sequence from [Ha 2], and prove the projective bundle formula. All these are refinements of the known formulas for Chow groups. For example if $f : X \to Y$ is a map of smooth quasi-projective varieties and $\alpha \in \mathrm{CH}^r(Y)$, we have $f^* \circ C(\alpha) = C(f^*\alpha) \circ f^*$ as morphisms $h(Y) \to h(X)(r)[2r]$. This reflects the identity $f^*(\alpha \cdot y) = f^*(\alpha) \cdot f^*(y)$ for $y \in \mathrm{CH}^*(Y)$.

In §3, after recalling the definition of the additive category of relative pure motives $\mathcal{CHM}(S)$, where S is a variety, we show the projective bundle, the self-intersection and the blow-up formulas in the relative setting. A typical object of $\mathcal{CHM}(S)$ is of the form $h(X/S)(r)$, where X is a smooth quasi-projective variety equipped with a projective map to S, and $r \in \mathbb{Z}$. For the prototype of these results, see [Ma] where the blow-up of smooth projective varieties is studied. Our results and proofs in this section are analogous to those in [Ma]; the identity principle and the split exact sequence principle play fundamental roles.

We also show that the association $h(X/S)(r) \mapsto \mathrm{CH}^{r+p}(X, n)$ is a partial functor. Using this we show the formulas (1.1) and (1.2) for higher Chow groups follow from the formulas for relative motives.

We naturally expect analogous self-intersection and blow-up formulas to hold for mixed motives. In § §4 and 5 we prove them. In §4 we show the functor $X \mapsto h(X)$ explained in §2 can be extended to a "partial" functor L from $\mathcal{CHM}(S)$ to $\mathcal{D}(k)$. Thus to each object M of $\mathcal{CHM}(S)$ there corresponds an object $L(M)$ of $\mathcal{D}(k)$; if $M = h(X/S)(r)$, then $L(M) = h(X)(r)[2r]$. In addition, if (M, N) is a pair of objects satisfying the condition of *admissibility*, and $u \in \mathrm{Hom}_{\mathcal{CHM}(S)}(M, N)$, there is an induced map $L(u) : L(M) \to L(N)$.

As explained in §5, if we take, say, the blow-up formula for relative motives, and apply the partial functor L, we obtain the blow-up formula for mixed motives.

At the end of §5 we indicate an alternative proof of the formulas for mixed motives. It proceeds by repeating the proof of [SGA], with Chow groups $\mathrm{CH}^r(X)$ replaced with motives $h(X)(r)$. One must use the formulas proven in §2. Since the argument is lengthy and not along the main line of this paper, we only write it down for the self-intersection formula.

We would like to thank the referee who carefully read the manuscript, gave us useful suggestions, and asked for clarification in (3.6).

§1. **The blow-up formula for higher Chow groups.**

In this paper we consider schemes over a field k.

We refer to [Bl 1] and [Bl 2] for the details of the theory of higher Chow groups. The following is a list of their properties we will use in this paper.

(1) Let $\square^1 = \mathbb{P}^1_k - \{1\}$ and $\square^n = (\square^1)^n$ with coordinates $(x_1, \cdots, x_n)$. Faces of $\square^n$ are intersections of codimension one faces, and the latter are divisors of the form $\square^{n-1}_{i,a} = \{x_i = a\}$ where $a = 0$ or ∞. A face of dimension m is canonically isomorphic to $\square^m$.

Let X be an equi-dimensional variety (or a scheme). Let $\mathcal{Z}^r(X \times \square^n)$ be the free abelian group on the set of codimension r irreducible subvarieties of $X \times \square^n$ meeting each $X \times$ face properly. An element of $\mathcal{Z}^r(X \times \square^n)$ is called an *admissible* cycle. The inclusions of codimension one faces $\delta_{i,a} : \square^{n-1}_{i,a} \hookrightarrow \square^n$ induce the map

$$\partial = \sum (-1)^i (\delta^*_{i,0} - \delta^*_{i,\infty}) : \mathcal{Z}^r(X \times \square^n) \to \mathcal{Z}^r(X \times \square^{n-1}) \ .$$

One has $\partial \circ \partial = 0$. Let $\pi_i : X \times \square^n \to X \times \square^{n-1}$, $i = 1, \cdots, n$ be the projections, and $\pi_i^* : \mathcal{Z}^r(X \times \square^{n-1}) \to \mathcal{Z}^r(X \times \square^n)$ be the pull-backs. Let $\mathcal{Z}^r(X, n)$ be the quotient of $\mathcal{Z}^r(X \times \square^n)$

by the sum of the images of π_i^*. Thus an element of $\mathcal{Z}^r(X,n)$ is represented uniquely by a cycle whose irreducible components are non-degenerate (not a pull-back by π_i). The map ∂ induces a map $\partial : \mathcal{Z}^r(X,n) \to \mathcal{Z}^r(X,n-1)$, and $\partial \circ \partial = 0$. The complex $\mathcal{Z}^r(X,\cdot)$ thus defined is the *cycle complex* of X in codimension r. The higher Chow groups are the homology groups of this complex:

$$\mathrm{CH}^r(X,n) = H_n \mathcal{Z}^r(X,\cdot) \ .$$

Note $\mathrm{CH}^r(X,0) = \mathrm{CH}^r(X)$, the Chow group of X. In this paper we would rather use the indexing by dimensions: for $s \in \mathbb{Z}$, $\mathcal{Z}_s(X,\cdot) = \mathcal{Z}^{\dim X - r}(X,\cdot)$, and $\mathrm{CH}_s(X,n)$ is the homology group.

(2) For a proper map $f : X \to Y$ of k-schemes, the push-forward $f_* : \mathcal{Z}_s(X,\cdot) \to \mathcal{Z}_s(Y,\cdot)$, hence also $f_* : \mathrm{CH}_s(X,n) \to \mathrm{CH}_s(Y,n)$ is defined.

(3) For a flat map $f : X \to Y$ of (relative) equi-dimension d, the pull-backs $f^* : \mathcal{Z}_s(Y,\cdot) \to \mathcal{Z}_{s+d}(X,\cdot)$ and $f^* : \mathrm{CH}_s(Y,n) \to \mathrm{CH}_{s+d}(X,n)$ are defined. If $f : X \to Y$ be a map where Y is smooth and X equi-dimensional, there is a map $f^* : \mathrm{CH}^r(Y,n) \to \mathrm{CH}^r(X,n)$. In fact there is a quasi-isomorphic subcomplex $\mathcal{Z}^r(Y,\cdot)'$ of $\mathcal{Z}^r(Y,\cdot)$ on which $f^* : \mathcal{Z}^r(Y,\cdot)' \to \mathcal{Z}^r(X,\cdot)$ is defined.

(4) If X is smooth quasi-projective and equi-dimensional, one has the intersection product $\mathrm{CH}_s(X,n) \otimes \mathrm{CH}_t(X,m) \to \mathrm{CH}_{s+t-\dim X}(X,n+m)$.

(5) Projection formula.

(6) Projective bundle formula.

(7) Localization sequence. If X is a quasi-projective variety and U is an open set, letting $Z = X - U$, one has an exact sequence of complexes $0 \to \mathcal{Z}_s(Z,\cdot) \to \mathcal{Z}_s(X,\cdot) \to \mathcal{Z}_s(U,\cdot)$. The localization theorem [Bl-2] asserts that the induced map $\mathcal{Z}_s(X,\cdot)/\mathcal{Z}_s(Z,\cdot) \to \mathcal{Z}_s(U,\cdot)$ is a quasi-isomorphism.

For the basic notions of intersection theory, see [Fu]. For a locally free sheaf of $\mathcal{O}$-modules of finite rank E on a scheme X, let $g : \mathbb{P}(E) = \operatorname{Proj}\operatorname{Sym}(E^\vee) \to X$ be the associated projective bundle. So there is a canonical surjection $g^* E^\vee \to \mathcal{O}(1)$.

We will show the self-intersection formula (1.1) and the blow-up formula (1.2) for higher Chow groups. For ordinary Chow groups, these are in [SGA] or [Fu, §6.7].

More precisely, for ordinary Chow groups, [SGA, 9.1-9.8] proves Theorem (1.1) and (a) of Theorem (1.2), and [*loc. cit.* 9.9] proves Theorem (1.2), (b), (c), (e) and (f). For higher Chow groups, reading $\mathrm{CH}_*(X,n)$ for $\mathrm{CH}_*(X)$ and changing nothing otherwise in [*loc. cit.* 9.1-9.8], one obtains the proof of Theorem (1.1) and Theorem (1.2), (a). (The argument is based only on the projection formula and the projective bundle formula.) The proof of the rest of Theorem (1.2) for $n \geq 0$, given below, is not the same as [*loc. cit.* 9.9], and one needs to show (1.2), (d) using the localization theorem.

(1.1) **Theorem** (Self-intersection formula). *Let X be a smooth quasi-projective variety and $Y \subset X$ a smooth closed subvariety of codimension d, $i : Y \to X$ the closed immersion, and $N = N_Y X$ the normal bundle. Then*

$$i^* i_*(y) = c_d(N) \cdot y$$

for $y \in \mathrm{CH}_k(Y,n)$. (In [SGA], N denotes the conormal sheaf, which is dual to the normal bundle.)

(1.2) **Theorem.** *Let Y be a smooth quasi-projective variety, $X \subset Y$ a closed smooth subvariety of codimension d. Let $f : \tilde{Y} \to Y$ be the blow-up of Y along X, $\tilde{X} = f^{-1}(X)$ the exceptional divisor, $g : \tilde{X} \to X$ the induced map, and $i : X \to Y$ and $j : \tilde{X} \to \tilde{Y}$ the closed immersions.*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{j} & \tilde{Y} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

Let $N = N_X Y$ denote the normal bundle of X in Y, and $E := g^* N / \mathcal{O}_N(-1)$ the excess bundle. Then

(a) For $x \in \mathrm{CH}_k(X, n)$, $f^* i_* x = j_*(c_{d-1}(E) \cdot g^* x)$.
(b) For $y \in \mathrm{CH}_k(Y, n)$, $f_* f^* y = y$.
(c) If $\tilde{x} \in \mathrm{CH}_k(\tilde{X}, n)$, $g_*(\tilde{x}) = j^* j_*(\tilde{x}) = 0$, then $\tilde{x} = 0$.
(d) There is an exact sequence

$$0 \to \mathrm{CH}_k(\tilde{X}, n) \xrightarrow{a} \mathrm{CH}_k(X, n) \oplus \mathrm{CH}_k(\tilde{Y}, n) \xrightarrow{b} \mathrm{CH}_k(Y, n) \to 0$$

where

$$a(\tilde{x}) = (g_* \tilde{x}, -j_* \tilde{x}) \ ,$$
$$b(x, \tilde{y}) = i_*(\tilde{x}) + f_*(\tilde{y}) \ .$$

(e) If $\tilde{y} \in \mathrm{CH}_k(\tilde{Y}, n)$ satisfies $f_* \tilde{y} = j^* \tilde{y} = 0$, then $\tilde{y} = 0$.
(f) There is an exact sequence

$$0 \to \mathrm{CH}_k(X, n) \xrightarrow{\alpha} \mathrm{CH}_k(\tilde{X}, n) \oplus \mathrm{CH}_k(Y, n) \xrightarrow{\beta} \mathrm{CH}_k(\tilde{Y}, n) \to 0$$

where

$$\alpha(x) = (c_{d-1}(E) \cdot g^* x, -i_*(x)) \ ,$$
$$\beta(\tilde{x}, y) = j_*(\tilde{x}) + f^* y \ .$$

A left inverse of α is given by $\gamma(\tilde{x}, y) = g_* \tilde{x}$.

Proof. (b) Obvious from the definition.
(c) By the projective bundle formula

$$\tilde{x} = \sum_{i=0}^{d-1} c_1(\mathcal{O}_N(1))^i \cdot g^* x_i$$

with $x_i \in \mathrm{CH}_{k-d+1+i}(X, n)$. Since $g_*(c_1(\mathcal{O}_N(1))^i) = 0$ for $i < d-1$ and $= [X]$ for $i = d-1$ (see [Fu, Proposition (3.1)]) one has

$$0 = g_*(\tilde{x}) = x_{d-1} \ .$$

Using this and the self-intersection formula for j,

$$0 = j^* j_*(\tilde{x}) = \sum_{i=0}^{d-2} c_1(\mathcal{O}_N(1))^{i+1} \cdot g^* x_i \ .$$

So $x_i = 0$ for all i, thus $\tilde{x} = 0$.

(d) From the localization sequences of $i : X \to Y$ and $j : \tilde{X} \to \tilde{Y}$, one deduces the long exact sequence

$$\to \mathrm{CH}_k(\tilde{X}, n) \xrightarrow{a} \mathrm{CH}_k(X, n) \oplus \mathrm{CH}_k(\tilde{Y}, n) \xrightarrow{b} \mathrm{CH}_k(\tilde{X}, n) \to \cdots \ .$$

By (c), the map a is injective. So the map b is surjective.

(e) Since $f_*(\tilde{y}) = 0$, by (d), there exists $\tilde{x}$ such that $\tilde{y} = j_*(\tilde{x})$ and $g_*(\tilde{x}) = 0$. Then $j^* j_*(\tilde{x}) = j^*(\tilde{y}) = 0$. By (c) $\tilde{x} = 0$; hence $\tilde{y} = 0$.

(f) For an arbitrary element $\tilde{y} \in \mathrm{CH}_k(\tilde{Y}, n)$, $z := \tilde{y} - f^* f_*(\tilde{y})$ satisfies $f_*(z) = 0$. As in (e), there is $\tilde{x}$ such that $z = j_*(\tilde{x})$. So $\tilde{y} = f^* f_*(\tilde{y}) + j_*(\tilde{x})$; hence the surjectivity of the map β.

(a) implies $\beta \circ \alpha = 0$. There remains the exactness in the middle. Suppose

$$j_*(\tilde{x}) + f^*(y) = 0 \ .$$

Then $y = -f_* j_*(\tilde{x}) = -i_* g_*(\tilde{x})$. Let

$$\tilde{x}' := \tilde{x} - c_{d-1}(E) \cdot g^* g_* \tilde{x} \ ;$$

then $g_*(\tilde{x}') = 0$ since $g_*(c_{d-1}(E) \cdot g^* g_* \tilde{x}) = g_*(\tilde{x})$, [Fu, Example 3.3.3]. We have

$$j_*(\tilde{x}') = j_*(\tilde{x}) - f^* i_*(g_* \tilde{x}) = j_*(\tilde{x}) + f_* y = 0 \ .$$

By (c), $\tilde{x}' = 0$, so $\tilde{x} = c_{d-1}(E) \cdot g^* g_* \tilde{x}$, thus $(\tilde{x}, y) = \alpha(g_* \tilde{x})$. □

(1.3) **Corollary.** *The map*

$$\beta : \mathrm{Ker}[\mathrm{CH}_k(\tilde{X}, n) \xrightarrow{g_*} \mathrm{CH}_k(X, n)] \oplus \mathrm{CH}_k(Y, n) \to \mathrm{CH}_k(\tilde{Y}, n)$$

is an isomorphism.

(1.4) **Corollary.** *Let $\mathcal{Z}^r(X)$ be the cycle complex of codimension r. The maps*

$$\begin{aligned} \mathcal{Z}^{r-1}(\tilde{X}) \oplus \mathcal{Z}^r(Y) &\to \mathcal{Z}^{r-d}(X) \oplus \mathcal{Z}^r(\tilde{Y}), \\ (\tilde{x}, y) \quad &\mapsto (g_* \tilde{x}, j_* \tilde{x} + f^* y) \end{aligned}$$

and

$$\begin{aligned} \mathrm{Ker}[\mathcal{Z}^{r-1}(\tilde{X}) \xrightarrow{g_*} \mathcal{Z}^{r-d}(X)] \oplus \mathcal{Z}^r(Y) &\to \mathcal{Z}^r(\tilde{Y}), \\ (\tilde{x}, y) \quad &\mapsto j_* \tilde{x} + f^* y \end{aligned}$$

are quasi-isomorphisms.

In the above, more precisely one has to replace $\mathcal{Z}^r(Y)$ by a quasi-isomorphic subcomplex in order for f^* to be defined. The same remark applies to the following statement, which is an important case where contravariant descent property for cycle complex holds. For further development, see [Ha 2].

(1.5) **Theorem.** *Under the same hypothesis the map*

$$(f^*, g^*) : \mathrm{Cone}[\mathcal{Z}^r(Y) \xrightarrow{i^*} \mathcal{Z}^r(X)] \to \mathrm{Cone}[\mathcal{Z}^r(\tilde{Y}) \xrightarrow{j^*} \mathcal{Z}^r(\tilde{X})]$$

is a quasi-isomorphism.

Proof. By the next lemma, the statement is equivalent to the map

$$(i^*, j^*) : \mathrm{Cone}\, f^* \to \mathrm{Cone}\, g^*$$

being an quasi-isomorphism. Since $f^* : \mathrm{CH}^r(Y, n) \to \mathrm{CH}^r(\tilde{Y}, n)$ and $g^* : \mathrm{CH}^r(X, n) \to \mathrm{CH}^r(\tilde{X}, n)$ are injective, one has to show the map

$$(i^*, j^*) : \mathrm{Cok}[f^* : \mathrm{CH}^r(Y, n) \to \mathrm{CH}^r(\tilde{Y}, n)] \to \mathrm{Cok}[g^* : \mathrm{CH}^r(X, n) \to \mathrm{CH}^r(\tilde{X}, n)]$$

is an isomorphism.

The following square is commutative:

$$\begin{array}{ccc}
\mathrm{Ker}[\mathrm{CH}^{r-1}(\tilde{X}, n) \xrightarrow{g_*} \mathrm{CH}^{r-d}(X, n)] \oplus \mathrm{CH}^r(Y, n) & \xrightarrow{(j_*, f^*)} & \mathrm{CH}^r(\tilde{Y}, n) \\
\downarrow{\scriptstyle id \oplus i^*} & & \downarrow{\scriptstyle j^*} \\
\mathrm{Ker}[\mathrm{CH}^{r-1}(\tilde{X}, n) \xrightarrow{g_*} \mathrm{CH}^{r-d}(X, n)] \oplus \mathrm{CH}^r(X, n) & \xrightarrow{(j^* j_*, g^*)} & \mathrm{CH}^r(\tilde{X}, n) .
\end{array}$$

The upper horizontal arrow is an isomorphism. So is the lower horizontal arrow by the self-intersection formula and the projective bundle formula. Hence the assertion follows. □

(1.6) **Lemma.** *Let*

$$\begin{array}{ccc}
X & \xrightarrow{v} & Y \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u'} \\
Z & \xrightarrow{v'} & W
\end{array}$$

be a commutative diagram of complexes of abelian groups. Then $(v, v') : \mathrm{Cone}\, u \to \mathrm{Cone}\, u'$ *is a quasi-isomorphism if and only if* $(u, u') : \mathrm{Cone}\, v \to \mathrm{Cone}\, v'$ *is a quasi-isomorphism.*

Proof. Easy and left to the reader. □

§2. The motives of smooth varieties.

We review the notion of a distinguished subcomplex of the cycle complex. See [Ha 1, Part II, §1] for the case of smooth projective varieties. The generalization to smooth quasi-projective varieties was communicated to us by M. Levine, and included in [Ha 1, Part I, §1].

Let X be a smooth quasi-projective variety. Let Y be another smooth quasi-projective variety and $W = \{W_i\}$ a finite set where W_i is an irreducible closed set of $X \times Y \times \square^{\ell_i}$ meeting faces properly. Let $\mathcal{Z}^r_W(X, \cdot)$ be the subcomplex of $\mathcal{Z}^r(X, \cdot)$ generated by irreducible cycles z in $\mathcal{Z}^r(X, n)$ satisfying the following condition: For each face $F \subset \square^n$,

$$z \times Y \times \square^{\ell_i} \quad \text{and} \quad W_i \times F \quad \text{meet properly in } X \times Y \times \square^{n+\ell_i} .$$

The inclusion of the subcomplex $\mathcal{Z}_W(X, \cdot) \hookrightarrow \mathcal{Z}(X, \cdot)$ is a quasi-isomorphism. A subcomplex of this form is called a *distinguished subcomplex*; it is simply written $\mathcal{Z}^r(X, \cdot)'$ when it is not necessary to specify W.

For X, Y smooth quasi-projective, let $\mathcal{Z}^r_{pr}(X, Y, \cdot)$ be the subcomplex of $\mathcal{Z}^r(X \times Y, \cdot)$ generated by irreducible subvarieties z whose support $|z|$ is proper over Y. As a consequence of the moving lemma, one has:

(1) For $f \in \mathcal{Z}^s_{pr}(X, Y, \ell)$, there is a distinguished subcomplex $\mathcal{Z}^r(X, \cdot)'$ such that the map of graded abelian groups

$$f_* : \mathcal{Z}^r(X, \cdot)' \to \mathcal{Z}^{r+s-\dim Y}(Y, \cdot + \ell) ,$$

$f_*(z) = p_{Y*}[(f\times\square^n)\cdot(z\times Y\times\square^\ell)]$, is defined. More generally for any T smooth quasi-projective, there is a distinguished subcomplex $\mathcal{Z}^r(T\times X,\cdot)'$ such that the map

$$f_* : \mathcal{Z}^r(T\times X,\cdot)' \to \mathcal{Z}^{r+s-\dim Y}(T\times Y,\cdot+\ell)\ ,$$

is defined. One has

$$(\partial f)_*(z) = \partial(f_*(z)) - (-1)^\ell f_*(\partial z)\ .$$

For $\ell = 0$, f_* is a map of complexes. If $f_1, f_2 \in \mathcal{Z}^s_{pr}(X,Y,0)$ and $f_1 - f_2 = \partial F$ for an element $F\in\mathcal{Z}^s_{pr}(X,Y,1)$, then $(f_1)_*$ and $(f_2)_*$ are homotopy equivalent.

(2) Let $f\in\mathcal{Z}^s_{pr}(X,Y,\ell)$, $g\in\mathcal{Z}^t_{pr}(Y,Z,m)$ be elements such that $\{f\times Z,\ X\times g\}$ is properly intersecting in $X\times Y\times Z$; then $g\circ f\in\mathcal{Z}^{s+t-\dim Y}_{pr}(X,Z,\ell+m)$ is defined. Let T be another smooth quasi-projective variety. We then have distinguished subcomplexes $\mathcal{Z}(T\times X,\cdot)'$, $\mathcal{Z}(T\times Y,\cdot)'$ such that the maps

$$f_* : \mathcal{Z}(T\times X,\cdot)' \to \mathcal{Z}(T\times Y,\cdot+\ell)'\ ,$$
$$g_* : \mathcal{Z}(T\times Y,\cdot)' \to \mathcal{Z}(T\times Z,\cdot+m)\ ,$$
$$(g\circ f)_* : Z(T\times X,\cdot)' \to \mathcal{Z}(T\times Z,\cdot+\ell+m)\ ,$$

are all defined and $(g\circ f)_* = g_*f_*$. This can be generalized to the case of a finite sequence of composable correspondences.

Let $\alpha\in\mathrm{CH}^r(X)$. Take a representative $\tilde\alpha\in\mathcal{Z}^r(X,0)$ of α, and consider

$$\delta_*(\tilde\alpha)\in\mathcal{Z}^{r+\dim X}(X\times X,0)$$

where $\delta : X\to X\times X$ is the diagonal embedding. Note $\delta_*(\tilde\alpha)\in\mathcal{Z}^{r+\dim X}_{pr}(X,X,0)$. We have the induced map

$$(\delta_*(\tilde\alpha))_* : \mathcal{Z}^s(X,\cdot)' \to \mathcal{Z}^{s+r}(X,\cdot)\ ;$$

its homotopy class is independent of the choice of a representative.

Let $\varphi : X\to Y$ be a map of smooth quasi-projective varieties. Its graph $\Gamma_\varphi\subset Y\times X$ is an element of $\mathcal{Z}^{\dim Y}_{pr}(Y,X,0)$, and induces the map

$$\varphi^* = (\Gamma_\varphi)_* : \mathcal{Z}^r(Y)' \to \mathcal{Z}^r(X)\ .$$

If φ is proper, taking the transpose one has a correspondence ${}^t\Gamma_\varphi\subset X\times Y\in\mathcal{Z}^{\dim Y}_{pr}(X,Y,0)$. It induces the map

$$\varphi_* = ({}^t\Gamma_\varphi)_* : \mathcal{Z}^r(X)\to\mathcal{Z}^{r-\dim X+\dim Y}(Y)\ .$$

We refer to [Ha 1] or [Ha 2, §4] for the details about the triangulated category of mixed motives $\mathcal{D}(k)$. In [Ha 1] we took the rational cycle complex and considered a $\mathbb{Q}$-linear category, but if we take the integral cycle complex as in §1, we obtain a $\mathbb{Z}$-linear category. (In this case, however, we do not have duals or internal Hom's.) For the purposes of this paper, we recall some definitions.

(1) A *finite symbol* K over k is a finite formal sum $\bigoplus_\alpha(X_\alpha,r_\alpha)$, where X_α is a smooth projective variety and r_α an integer. One has the direct sum and the tensor product for finite symbols: $(X,r)\otimes(X',r') = (X\times X',r+r')$. For finite symbols K, K', one has a complex of abelian groups $\mathrm{Hom}(K,K')^\bullet$. If $K=(X,r)$ and $K'=(Y,s)$, then

$$\mathrm{Hom}((X,r),(Y,s)\,)^\bullet = \mathcal{Z}^{\dim X+s-r}(X\times Y,-\bullet)\ .$$

For finite symbols K, K' and K'', there is a partially defined, associative composition map

$$\mathrm{Hom}(K,K')^\bullet \otimes \mathrm{Hom}(K',K'')^\bullet \dashrightarrow \mathrm{Hom}(K,K'')^\bullet \ .$$

(2) The category $\mathcal{D}(k)$ is a pseudo-abelian triangulated category with tensor product. It is the pseudo-abelianization of a slightly smaller triangulated tensor category $\mathcal{D}_{\mathrm{finite}}(k)$ which we now describe. An object of $\mathcal{D}_{\mathrm{finite}}(k)$ is of the form $K = (K^m; f^{m,n})$ where K^m are finite symbols indexed by $m \in \mathbb{Z}$, almost all of which being zero, and $f^{m,n} \in \mathrm{Hom}(K^m, K^n)^{-n+m+1}$, $m < n$, are elements satisfying the condition

$$(-1)^n \partial f^{m,n} + \sum_{m<\ell<n} f^{\ell,n} \circ f^{m,\ell} = 0 \ .$$

For objects K, L in $\mathcal{D}_{\mathrm{finite}}(k)$, one has a complex of abelian groups $\mathrm{Hom}(K,L)^\bullet$, generalizing $\mathrm{Hom}(K,L)^\bullet$ for finite symbols. For three objects K, L, M, there is a partially defined composition map

$$\mathrm{Hom}(K,L)^\bullet \otimes \mathrm{Hom}(L,N)^\bullet \dashrightarrow \mathrm{Hom}(K,L)^\bullet$$

defined on a quasi-isomorphic subcomplex. The homomorphism group in $\mathcal{D}_{\mathrm{finite}}(k)$ is defined by

$$\mathrm{Hom}_{\mathcal{D}_{\mathrm{finite}}(k)}(K,L) = H^0\,\mathrm{Hom}(K,L)^\bullet \ ,$$

and the composition of morphisms induced from the above composition map by taking the 0-th cohomology.

It is a theorem that $\mathcal{D}_{\mathrm{finite}}(k)$ thus defined has the structure of a tensor triangulated category (see [Ha 1, II, §4]).

We have the Tate object $\mathbb{Z}(r)$ defined as (pt, r) placed in degree $2r$. Thus for K in $\mathcal{D}_{\mathrm{finite}}(k)$, the Tate twist $K(r) = K \otimes \mathbb{Z}(r)$ is defined. Specifically, $K \otimes \mathbb{Z}(r)$ is the object $(L; g^{m,n})$, where

$$L^m = K^{m-2r} \otimes (pt, r) \ ,$$

$$g^{m,n} = f^{m-2r,n-2r} \in \mathrm{Hom}(L^m, L^n)^\bullet = \mathrm{Hom}(K^{m-2r}, K^{n-2r})^\bullet \ .$$

So $K(r)[2r]$ is the object $(K^m \otimes (pt,r); f^{m,n})$.

In this paper we work in $\mathcal{D}_{\mathrm{finite}}(k)$, so write it simply $\mathcal{D}(k)$.

(4) There is a natural functor from the category of smooth projective varieties

$$h : (\textit{Smooth Proj}\,/k)^{opp} \to \mathcal{D}(k) \ ,$$

which takes X to $(X, 0)$ placed in degree 0. There is a functor

$$\mathrm{CH}^r(-,n) = \mathrm{Hom}_{\mathcal{D}(k)}(\mathbb{Z}(0), (-)(r)[2r-n]) : \mathcal{D}(k) \to (Ab) \ ,$$

so that for X smooth projective $\mathrm{CH}^r(h(X), n) = \mathrm{CH}^r(X, n)$.

We recall the definition of the functor of *cohomological motives* for smooth quasi- projective varieties. This is mostly taken from [Ha 2, §5]. For the rest of this paper we assume the characteristic of k is zero. Let *(Smooth Q-Proj/k)* denote the category of smooth quasi-projective varieties over k.

For a smooth projective variety T, $t \in \mathbb{Z}$, X in *(Smooth Q-Proj/k)* and $r \in \mathbb{Z}$ define a complex

$$H((T,t),(X,r))^\bullet := \mathcal{Z}_{\dim X - r + t}(T \times X, -\bullet)$$

("correspondences" from (T,t) to (X,r)). For an object $K \in \mathcal{D}(k)$ define

$$H(K,(X,r))^\bullet = Tot\,(K^m,(X,r))^\bullet \ ,$$

where the right hand side is the total complex of a collection of complexes, defined in [Ha 2, §4]. Set $H(K,(X,r)) = H^0 H(K,(X,r))^\bullet$, the 0-th cohomology. When $r=0$, we also write $H(K,X)$ for $H(K,(X,0)\,)$. We have a partially defined map (K' another object of $\operatorname{Ob}\mathcal{D}(k)$)

$$\begin{array}{ccc} \operatorname{Hom}(K',K)^\bullet \otimes H(K,(X,r))^\bullet & - - \to & H(K',(X,r))^\bullet \\ v \otimes \alpha & \mapsto & \alpha \circ v \ . \end{array}$$

It can be shown this map is defined on quasi-isomorphic subcomplexes by an argument similar to [Ha 2, II, §1]. Passing to cohomology one has

$$\operatorname{Hom}_{\mathcal{D}_{finite}(k)}(K',K) \otimes H(K,(X,r)) \to H(K',(X,r)) \ .$$

We have the following functoriality in v, both at the chain level and on cohomology: $\alpha\circ(v\circ v') = (\alpha\circ v)\circ v'$.

For X, Y smooth quasi-projective, let

$$\operatorname{Hom}((X,r),(Y,s)\,)^\bullet = \mathcal{Z}_{pr}^{\dim X+s-r}(X,Y,-\bullet) \ .$$

For T smooth projective and $f \in \operatorname{Hom}((X,r),(Y,s)\,)^\ell$, one has the map

$$f_* : H((T,t),(X,r)\,)^\bullet \to H((T,t),(Y,s)\,)^{\bullet+\ell} \ ;$$

here we take an appropriate distinguished subcomplex of $H((T,t),(X,r)\,)^\bullet$ denoted by the same notation. Thus for an object $K \in \mathcal{D}(k)$ we have

$$f_* : H(K,(X,r)\,)^\bullet \to H(K,(Y,s)\,)^{\bullet+\ell} \ .$$

One has $(\partial f)_* = \partial\circ f_* - (-1)^\ell f_*\circ\partial$. If $g \in \operatorname{Hom}((Y,s),(Z,t)\,)^m$ and $g\circ f$ is defined, then the maps g_*, $(g\circ f)_*$ are also defined and one has the identity $g_* f_* = (g\circ f)_*$. Here one must take appropriate quasi-isomorphic subcomplexes of $H(K,(X,r)\,)^\bullet$ and $H(K,(Y,s)\,)^\bullet$. The same for a composable sequence of correspondences.

If $f \in \operatorname{Hom}((X,r),(Y,s)\,)^0$, then f_* is a map of complexes, so there is an induced map on cohomology, denoted by the same $f_* : H(K,(X,r)\,) \to H(K,(Y,s)\,)$. It depends only on the class $[f] \in H^0\operatorname{Hom}((X,r),(Y,s)\,)^\bullet$. The following associativity holds, both at the chain level and on cohomology:

$$\alpha\circ(v\circ v') = (\alpha\circ v)\circ v' \quad \text{if } v \text{ and } v' \text{ are compolable,}$$

$$(f_*\alpha)\circ v = f_*(\alpha\circ v) \ .$$

(2.1) **Definition.** Let $X \in \operatorname{Ob}\,(\mathit{Smooth\ Q\text{-}Proj}/k))$ and $r \in \mathbb{Z}$. A pair (L,α) where $L \in \mathcal{D}(k)$, $\alpha \in H(L,(X,r))$ is a *left resolution* of (X,r) if for any $K \in \mathcal{D}(k)$ the map

$$\alpha\circ(-) : \operatorname{Hom}_{\mathcal{D}(k)}(K,L) \to H(K,(X,r))$$

is an isomorphism. A left resolution is unique up to unique isomorphism.

(2.2) **Theorem.** *(1) For any object X of (Smooth Q-Proj$/k$) and $r \in \mathbb{Z}$, its left resolution $L((X,r))$ exists. When $r = 0$, we write $L(X)$ or $h(X)$ for $L((X,0))$. One has $L((X,r)) = L(X)(r)[2r]$.*

(2) For $u \in H^0 \operatorname{Hom}((X,r),(Y,s)\,)$, there is a unique morphism $L(u) : L((X,r)) \to L((Y,s))$ such that the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{D}(k)}(K, L((X,r)) & \overset{\sim}{\to} & H(K,(X,r)\,) \\ {\scriptstyle L(u)}\downarrow & & {\scriptstyle u_*}\downarrow \\ \operatorname{Hom}_{\mathcal{D}(k)}(K, L((Y,s)) & \overset{\sim}{\to} & H(K,(Y,s)\,) \end{array}$$

commutes. If elements $u \in \operatorname{Hom}((X,r),(Y,s)\,)^0$ and $v \in \operatorname{Hom}((Y,s),(Z,t)\,)^0$ are composable, then $L(v \circ u) = L(v)L(u)$.

(3) For $\alpha \in \mathrm{CH}^r(X)$ there is the associated map $C_\alpha : h(X) \to h(X)(r)[2r]$. For a proper map $f : X \to Y$ of smooth quasi-projective equi-dimensional varieties, there is the associated map $f_* : h(X)(\dim X)[2\dim X] \to h(Y)(\dim Y)[2\dim Y]$. The association $X \mapsto L(X)$ uniquely extends to a functor

$$h : (\text{Smooth Q-Proj}/k)^{opp} \to \mathcal{D}(k)$$

such that the isomorphism

$$\operatorname{Hom}_{\mathcal{D}(k)}(K, L(X)) \to H(K,X)$$

is contravariantly functorial in X. Similarly $X \mapsto L(X)(\dim X)[2\dim X]$ extends to a functor

$$h' : (\text{Smooth Q-Proj}/k;\ \text{proper}) \to \mathcal{D}(k)$$

such that the isomorphism

$$\operatorname{Hom}_{\mathcal{D}(k)}(K, L(X)(\dim X)[2\dim X]) \to H(K,(X,\dim X)\,)$$

is covariantly functorial for proper maps in X.

Remark. We call $h(X) = L(X)$ the cohomological motive of X.

For $f : X \to Y$, $f^* = h(f) : h(Y) \to h(X)$ induces the pull-back $f^* : \mathrm{CH}^r(Y,n) \to \mathrm{CH}^r(X,n)$ under the functor $\mathrm{CH}^r(-,n)$. The C_α in (2) induces the multiplication by α: $\mathrm{CH}^p(X,n) \to \mathrm{CH}^{r+p}(X,n)$. The f_* induces the push-forward $f_* : \mathrm{CH}_s(X,n) \to \mathrm{CH}_s(Y,n)$.

Proof. (1) Let $X \in$ *(Smooth Q-Proj$/k$)*, irreducible, and take its smooth compactification, namely an open immersion $j : X \to \bar{X}$ where $\bar{X}$ is smooth projective and $D = \bar{X} - X$ is a divisor with normal crossings. Let $D^{(i)}$ be the i-fold intersection of the components of D so one has a strict simplicial variety augmented to $\bar{X}$

$$\cdots \begin{smallmatrix}\to\\\to\\\to\end{smallmatrix} D^{(1)} \underset{d_1}{\overset{d_0}{\rightrightarrows}} D^{(0)} \to \bar{X}\ .$$

Taking its associated diagram (take the alternating sum of the transposes of the graphs of the face maps) one obtains an object of $\mathcal{D}(k)$

$$(\bar{X}\&D) := [\cdots \to (D^{(1)},-2) \to (D^{(0)},-1) \to (\bar{X},0)]$$

where $\bar{X}$ in degree 0. It follows from the localization theorem that for any T smooth projective and any s, the restriction

$$j^* : \mathcal{Z}_s(T \times (\bar{X}\&D)) \longrightarrow \mathcal{Z}_s(T \times X)$$

is a quasi-isomorphism. So if we define $\alpha = [\Gamma_j]$ (the graph of j) then $((\bar{X}\&D), \alpha)$ is a left resolution of X.

Since $H(K, (X, r)\,)^\bullet = H(K(-r)[-2r], (X, 0)\,)^\bullet$, if $L(X)$ and $\alpha \in H(L(X), (X, 0)\,)$ is a left resolution of X, then $L(X)(r)[2r]$ and $\alpha \in H(L(X)(r)[2r], (X, r)\,)$ is a left resolution of (X, r).

(2) is obvious from the definitions and the Yoneda lemma.

(3) follows from (2) applied to $\delta_*(\alpha)$, Γ_f, and ${}^t\Gamma_f$. □

We often write $h(X)((r))$ instead of $h(X)(r)[2r]$. The morphism C_α is also denoted by $C(\alpha)$, or just α.

(2.3) **Proposition.** *(1) For $\alpha \in \mathrm{CH}^r(X)$ and $\beta \in \mathrm{CH}^s(X)$,*

$$C(\beta) \circ C(\alpha) = C(\alpha \cdot \beta) : h(X) \to h(X)((r+s)) \; . \tag{2.3.1}$$

(2) For a proper map $f : X \to Y$ and $\alpha \in \mathrm{CH}^r(X)$, one has

$$f_* \circ C(\alpha) \circ f^* = C(f_*\alpha) : h(Y) \to h(X)((r - \dim X + \dim Y)) \; . \tag{2.3.2}$$

Here $f_*\alpha \in \mathrm{CH}^{r-\dim X+\dim Y}(Y)$.

(3) For a proper map $f : X \to Y$ and $\alpha \in \mathrm{CH}^r(Y)$,

$$f_* \circ C(f^*\alpha) = C(\alpha) \circ f_* : h(X) \to h(Y)((r - \dim Y + \dim X)) \; . \tag{2.3.3}$$

(4) For a map $f : X \to Y$ and $\alpha \in \mathrm{CH}^r(Y)$,

$$f^* \circ C(\alpha) = C(f^*\alpha) \circ f^* : h(Y) \to h(X)((r)) \; . \tag{2.3.4}$$

(5) Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

be a Cartesian diagram of equi-dimensional smooth quasi-projective varieties such that g, g' are closed immersions with the same codimension d. Then

$$f^* \circ g_* = g'_* \circ f'^* : h(Y') \to h(X)((d)) \; . \tag{2.3.5}$$

(6) Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g'}\uparrow & & \uparrow{\scriptstyle g} \\ X' & \xrightarrow{f'} & Y' \end{array}$$

be a Cartesian diagram of equi-dimensional smooth quasi-projective varieties such that f, f' are proper and g, g' are open immersions. Then

$$f_* \circ g'^* = g^* \circ f_* \; . \tag{2.3.6}$$

Proof. In each case the proof is reduced to the equality of the cycles representing the two sides of the identity. We show (1) and (2) to illustrate the method.

For (1), take representatives $\tilde{\alpha} \in \mathcal{Z}^r(X,0)$ and $\tilde{\beta} \in \mathcal{Z}^s(X,0)$ for α, β, which meet properly in X. Then $\delta_*(\tilde{\alpha}) \times X$ and $X \times \delta_*(\tilde{\beta})$ meet properly in $X \times X \times X$, and $\delta_*(\tilde{\beta}) \circ \delta_*(\tilde{\alpha}) = \delta_*(\tilde{\alpha} \cdot \tilde{\beta})$. This shows (1).

For (2), let $\tilde{\alpha} \in \mathcal{Z}^r(Y,0)$ be a representative of α such that $f^*\tilde{\alpha} \in \mathcal{Z}^r(X,0)$ is defined. Then $\delta_{X*}(f^*\tilde{\alpha}) \in \operatorname{Hom}((X,0),(X,r)\,)^0$ represents $C(f^*\alpha)$. Recall ${}^t\Gamma_f \in \operatorname{Hom}((X,\dim X),(Y,\dim Y)\,)^0$ represents f_*. The composition

$$ {}^t\Gamma_f \circ \delta_{X*}(f^*\tilde{\alpha}) \in H((X,0),(Y,r-\dim X+\dim Y)\,)^0 $$

is defined and equal to $({}^t\gamma_f)_*(f^*\tilde{\alpha})$. Here ${}^t\gamma_f : X \hookrightarrow X \times Y$ is (the transpose of) the graph of f. So $f_* \circ C(f^*\alpha)$ is represented by $({}^t\gamma_f)_*(f^*\tilde{\alpha})$. On the other hand the composition $\delta_{Y*}(\tilde{\alpha}) \circ {}^t\Gamma_f$ is also defined, and also equal to $({}^t\gamma_f)_*(f^*\tilde{\alpha})$. $\square$

(2.4) **Proposition.** *Let X be a smooth quasi-projective variety, $i : Z \hookrightarrow X$ a smooth closed subvariety of codimension d, and $j : U = X - Z \to X$ the open immersion of the complement. Then there is a distinguished triangle of the form*

$$ h(Z)((-d)) \xrightarrow{i_*} h(X) \xrightarrow{j^*} h(U) \xrightarrow{[1]} . $$

This follows from [Ha 2, Theorem (2.9)].

(2.5) **Proposition.** *Let X be a smooth quasi-projective variety. For E a locally free sheaf of rank $r+1$, one has $p : P = \mathbb{P}(E) \to X$ the associated projective bundle, and $\xi = c_1(\mathcal{O}(1)) \in$ $\mathrm{CH}^1(X)$. Then the morphism*

$$ \sum_{0 \le i \le r} C(\xi^i) \circ p^* : \bigoplus_{i=0}^{r} h(X)((-i)) \to h(P) $$

is an isomorphism.

Proof. We show $\sum_{0\le i\le r} C(\xi^i) \circ p^*$ is an isomorphism, its inverse being given by $(p_* \circ C(\xi^{r-i}))_{i=0,\cdots,r}$. That it gives a left inverse follows from:

$$ p_* \circ C(\xi^i) \circ p^* = C(p_*\xi^i) = \begin{cases} [X] & \text{if } i = r\ , \\ 0 & \text{if } i \neq r\ . \end{cases} $$

To show $\sum_{0\le i\le r} C(\xi^i) \circ p^*$ is an isomorphism we proceed by induction on $\dim X$; using the preceding proposition reduce to the case of the trivial bundle $P = X \times \mathbb{P}^r$. Then one can verify

$$ \sum_{i=0}^{r} C(\xi^i) \circ p^* \circ p_* \circ C(\xi^{r-i}) = id_{h(P)} \tag{2.5.1} $$

as follows. If $H^i \subset \mathbb{P}^r$ is a codimension r subspace, ξ^i is represented by $X \times H^i \subset X \times \mathbb{P}^r = P$. The composition

$$ \delta_*(X \times H^i) \circ \Gamma_p \circ {}^t\Gamma_p \circ \delta_*(X \times H^{r-i}) $$

equals $X \times H^{r-i} \times H^i$, so (2.5.1) is represented by $\sum X \times H^i \times H^{r-i}$ on $P \times_X P \subset P \times P$. Since $\sum H^{r-i} \times H^i \sim \Delta_{\mathbb{P}^r}$ (rational equivalence on $\mathbb{P}^r \times \mathbb{P}^r$), one has

$$\sum X \times H^{r-i} \times H^i \sim \Delta_P$$

in $\mathcal{Z}_{pr}(P, P, 0)$, hence follows (2.5.1). $\square$

§3. **The blow-up formula for relative motives.**

We will state and prove the projective bundle formula and the blow-up formula in the relative setting. It is then shown the association $(X/S, r) \mapsto \mathrm{CH}^{p+r}(X, n)$ is a partial functor. From this we derive the projective bundle formula and the blow-up formula for higher Chow groups, reproving the results in §1.

Let S be a quasi-projective variety over k. There is the theory of Chow motives over S, generalizing Chow motives over $\operatorname{Spec} k$ as in [Ma]. For the details see [CH]. In this section we consider ordinary (integral) Chow groups as in [Fu].

The category of Chow motives over S is denoted by $\mathcal{CHM}(S)$. It is a pseudo-abelian category. A typical object is of the form $(X/S, r)$ where X is a smooth variety with a projective map to S and r is an integer. Morphisms between such objects are

$$\mathrm{Hom}((X/S, r), (Y/S, s)) = \bigoplus_j \mathrm{CH}_{\dim Y_j - s + r}(X \times_S Y_j)$$

where Y_j are the components of Y. Composition of morphisms can be adequately defined, which we will not recall here. One must add images of projectors of objects as above to arrive at a pseudo-abelian category. We let $h(X/S)(r) = (X/S, r)$.

For convenience write *(Smooth Q-Proj ; proj*/S) for the category of smooth quasi-projective varieties equipped with projective maps to S. If X and Y are in *(Smooth Q-Proj ; proj*/S) and $f : X \to Y$ is an S-morphism, there is an induced morphism

$$f^* : h(Y/S) \to h(X/S)$$

and if moreover X and Y are equi-dimensional there is a morphism

$$f_* : h(X/S)(\dim X) \to h(Y/S)(\dim Y) \ .$$

For $\alpha \in \mathrm{CH}^r(X)$ let $C_\alpha = \delta_*(\alpha) \in \mathrm{CH}_{\dim X - r}(X \times_S X)$ where $\delta : X \to X \times_S X$ is the diagonal; it gives a map C_α, or just α, from $h(X/S)$ to $h(X/S)(r)$.

One states an identity principle (the proof is obvious).

(3.1) **Proposition.** *Let*

$$u : (X/S, r) \to (Y/S, s)$$

be a morphism in $\mathcal{CHM}(S)$. *It is zero if and only if the induced map under* $\mathrm{Hom}((Z/S, i), -)$, *for each* $(Z/S, i)$

$$u_* : \mathrm{CH}_{\dim X + i - r}(Z \times_S X) \to \mathrm{CH}_{\dim Y + i - s}(Z \times_S Y)$$

is zero.

The following fact (3.2) is [Ma, Proposition in §5].

(3.2) **Proposition.** *Let $\mathcal{D}$ be a pseudo-abelian category,*

$$Y \underset{c}{\overset{a}{\rightleftarrows}} X \overset{b}{\to} Z \tag{3.2.1}$$

be objects and morphisms in $\mathcal{D}$ such that $ca = id_Y$ and for any object T the sequence

$$0 \to \mathrm{Hom}_{\mathcal{D}}(T, Y) \xrightarrow{a\circ(-)} \mathrm{Hom}_{\mathcal{D}}(T, X) \xrightarrow{b\circ(-)} \mathrm{Hom}_{\mathcal{D}}(T, Z) \to 0$$

is exact. Then the sequence (3.2.1) is split exact, i.e. isomorphic to

$$Y \xrightarrow{(id_Y, 0)} Y \oplus Z \xrightarrow{p_2} Z \ .$$

(3.3) **Theorem.** *Let X be a smooth quasi-projective variety with a projective map to S, E a locally free sheaf of rank $r+1$ on X, and $\mathbb{P}(E)$ the associated projective bundle. Then there is a canonical isomorphism in $CH\mathcal{M}(S)$*

$$h(\mathbb{P}(E)/S) = h(X/S) \oplus h(X/S)(-1) \oplus \cdots \oplus h(X/S)(-r) \ .$$

Proof. Let $P = \mathbb{P}(E)$, $\pi : P \to X$ the projection, and $h = c_1(\mathcal{O}(1)) \in \mathrm{CH}^1(P)$. Let, for $j = 0, \cdots, r$,

$$\varphi_j = C_{h^j} \circ \pi^* : h(X/S)(-j) \to h(P/S) \text{ , and}$$
$$\psi_j : \pi_* \circ C_{h^{r-j}} : h(P/S) \to h(X/S)(-j) \ .$$

One has the identities

$$\psi_j \circ \varphi_j = id \quad \text{and} \quad \sum_j \varphi_j \circ \psi_j = id \ .$$

To verify this, by the identity principle, one reduces to the projective bundle formula for Chow groups for the projective bundle $P \times_S Z \to Z$. Thus $\varphi = \varphi_0 + \cdots + \varphi_r$ and $\psi = (\psi_0, \cdots, \psi_r)$ give mutually inverse isomorphisms between $h(X/S) \oplus \cdots \oplus h(X/S)(-r)$ and $h(P/S)$.

To state the blow-up sequence, in the situation of Theorem (1.2), we further assume Y is equipped with a projective map to a quasi-projective variety S. Then the varieties Y, X, $\tilde{Y}$ and $\tilde{X}$ may be viewed as relative motives $h(Y/S)$, etc.

(3.4) **Theorem.** *There is a split exact sequence in $CH\mathcal{M}(S)$*

$$h(X/S)(-d) \xrightarrow{\alpha} h(\tilde{X}/S)(-1) \oplus h(Y/S) \xrightarrow{\beta} h(\tilde{Y}/S)$$

where

$$\alpha = (c_{d-1}(E) \circ g^*, -i_*), \quad \beta = j_* + f^* \ .$$

A left inverse of α is given by $\gamma = g_*$.

Proof. If $f : X \to Y$ is a map over S and $f^* : h(Y/S) \to h(X/S)$, then the map it induces $\mathrm{CH}_{\dim Y+i}(Z \times_S Y) \to \mathrm{CH}_{\dim X+i}(Z \times_S X)$ coincides with the refined Gysin map $f^!$ (see [Fu, §6]). We leave the proof of this fact to the reader.

In order to prove the theorem, we have to note that the blow-up formula for ordinary Chow groups holds universally in the following sense. Let $Y' \to Y$ be a map from a not necessarily smooth variety Y'. By base change one obtains a Cartesian square

$$\begin{array}{ccc} \tilde{X}' & \xrightarrow{j'} & \tilde{Y}' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle f'} \\ X' & \xrightarrow{i'} & Y' \end{array}$$

Then all the statements in Theorem (1.2) holds for the ordinary Chow groups ($n = 0$) if one replaces f, g, i, j by f', g', i', j' respectively, E by its pull-back E' to X', g^* by g'^*, and f^* by $f^!$ (the refined Gysin map). In particular one has an exact sequence

$$0 \to \mathrm{CH}_k(X') \xrightarrow{\alpha} \mathrm{CH}_k(\tilde{X}') \oplus \mathrm{CH}_k(Y') \xrightarrow{\beta} \mathrm{CH}_k(\tilde{Y}') \to 0$$

where

$$\alpha(x) = (c_{d-1}(E') \cdot g'^* x, -i'_*(x)), \quad \beta(\tilde{x}, y) = j'_*(\tilde{x}) + f^! y \ ,$$

with left inverse of α given by $\gamma(\tilde{x}, y) = g'_* \tilde{x}$. The proof is the same as in [Fu, §6].

Thus for each $(Z/S, i)$ in $\mathcal{CHM}(S)$, the induced sequence

$$\begin{aligned} 0 \to \mathrm{Hom}((Z/S, i), h(X/S)(-d)) \to \mathrm{Hom}((Z/S, i), h(\tilde{X}/S)(-1) \oplus h(Y/S)) \\ \to \mathrm{Hom}((Z/S, i), h(\tilde{Y}/S)) \to 0 \end{aligned}$$

is exact. The identity $\gamma \circ \alpha = id$ also is verified using the identity principle and reducing to the corresponding identity for Chow groups. □

(3.5) **Corollary.** *There is a canonical isomorphism in $\mathcal{CHM}(S)$*

$$h(\tilde{Y}/S) = h(Y/S) \oplus \bigoplus_{i=1}^{d-1} h(X/S)(-i) \ .$$

This follows from (3.3) and (3.4).

We also have the self-intersection formula in the relative setting; this is related to (1.1) for ordinary Chow groups.

To state the self-intersection formula, in the situation of (1.1) assume X is equipped with a projective map to S. The we have:

(3.6) **Theorem.** *We have the identity*

$$i^* \circ i_* = C(c_d(N)) : h(Y/S) \to h(Y/S)(d) \ . \tag{3.6.1}$$

Proof. The proof of this is similar to that of (3.4), using that (1.1) for ordinary Chow groups holds universally, in the sense we specify below.

We apply the functor $\mathrm{Hom}((Z/S,i),-)$ and verify the two induced maps coincide. The outline is as follows.

For an element $\alpha \in \mathrm{CH}^r(Y)$, let $C(\alpha) : h(Y/S) \to h(Y/S)(r)$ be the corresponding map. For each $(Z/S,i)$ in $\mathcal{CHM}(S)$, the induced map

$$C(\alpha)\circ(-) : \mathrm{CH}_{\dim Y+i}(Z\times_S Y) \to \mathrm{CH}_{\dim Y-r+i}(Z\times_S Y)$$

coincides with $g^*(\alpha)(-)$. Here $g : Z\times_S Y \to Y$ is the projection and $g^*(\alpha)\in A^r(Z\times_S Y)$ (the latter is Chow cohomology, [Fu, §17.3]). Concretely $g^*(\alpha)$ is the collection of maps

$$g^*(\alpha)(-) : \mathrm{CH}_*(Z\times_S Y) \to \mathrm{CH}_{*-r}(Z\times_S Y)$$

given by $g^*(\alpha)(u) = \gamma^*(u\times\alpha)$, where $\gamma : Z\times_S Y \to (Z\times_S Y)\times Y$ is the graph of g (since γ is a regular embedding the pull-back γ^* is defined). The proof is straightforward. If $\alpha = c_r(E)$, the Chern class of a vector bundle, then $g^*(\alpha) = c_r(g^*E)$.

On the other hand consider any fiber square

$$\begin{array}{ccc} Y' & \xrightarrow{i'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{i} & X \end{array}$$

over i. For $y'\in \mathrm{CH}_*(Y')$ we have $i^! i'_*(y') = c_d(g^*N)\cap y'$. This variant of the self-intersection formula follows from [Fu, Corollary (6.3)], by an argument analogous to the one following [Fu, Corollary (6.3)].

In particular for the fiber square

$$\begin{array}{ccc} Z\times_S Y & \xrightarrow{i'} & Z\times_S X \\ {\scriptstyle g}\downarrow & & \downarrow \\ Y & \xrightarrow{i} & X \end{array}$$

and $u\in \mathrm{CH}_*(Z\times_S Y)$, one has $i^! i'_*(u) = c_d(g^*N)\cap u$

Since the map $i^*\circ i_*$ induces the map $i^! i'_*$ on Chow groups upon applying the functor $\mathrm{Hom}((Z/S,i),-)$, we obtain the assertion. $\square$

(3.7) Let $(X/S,r)$, $(Y/S,s)$ be a pair of objects in $\mathcal{CHM}(S)$. We say the pair is *admissible* if $X\times_S Y$ is smooth over k (the projections to X, Y need not be smooth). For such a pair we define a map, for each p,n,

$$\mathrm{Hom}_{\mathcal{CHM}(S)}((X/S,r),(Y/S,s)) \to \mathrm{Hom}(\mathrm{CH}^p(X,n),\mathrm{CH}^{p+s-r}(Y,n)),\quad u\mapsto u_*\ ,$$

where u_* is the composition

$$\begin{aligned} \mathrm{CH}^p(X,n) &\xrightarrow{pr_X^*} \mathrm{CH}^p(X\times_S Y,n) \xrightarrow{u} \oplus_j \mathrm{CH}_{\dim Y_j-s+r-p}(X\times_S Y_j,n) \\ &\xrightarrow{pr_{Y*}} \oplus_j \mathrm{CH}_{\dim Y_j-s+r-p}(Y_j,n) = \mathrm{CH}^{p+s-r}(Y,n)\ . \end{aligned}$$

The map $u\mapsto u_*$ is additive.

A triple $(X/S,r)$, $(Y/S,s)$, $(Z/S,t)$ of objects is said to be admissible if the products $X\times_S Y$, $Y\times_S Z$, $X\times_S Z$, and $X\times_S Y\times_S Z$ are all smooth. Then for $u\in \mathrm{Hom}_{\mathcal{CHM}(S)}((X/S,r),(Y/S,s))$ and $v\in \mathrm{Hom}_{\mathcal{CHM}(S)}((Y/S,s),(Z/S,t))$, the composition

$$v\circ u\in \mathrm{Hom}_{\mathcal{CHM}(S)}((X/S,r),(Z/S,t))$$

is defined by the usual formula $v\circ u=p_{13*}(p_{12}^*u\cdot p_{23}^*v)$, where for example $p_{12}: X\times_S Y\times_S Z\to X\times_S Y$ is the projection. It is easy to verify $(v\circ u)_*=v_*\circ u_*$.

This can be generalized to the case of more than three objects $(X_1/S,r_1),\cdots,(X_n/S,r_n)$ and $u_i\in \mathrm{Hom}_{\mathcal{CHM}(S)}((X_i,r_i),(X_{i+1},r_{i+1}))$, $i=1,\cdots,n-1$. One can further extend this by linearity to a sequence of objects $M_1,\cdots,M_n$ of $\mathcal{CHM}(S)$.

(3.8) We give an alternative proof of Theorem (1.2), (a) and (f). We take $S=Y$ in (3.4), and get a split exact sequence in $\mathcal{CHM}(Y)$

$$h(X/Y)(-d)\xrightarrow{\alpha} h(\tilde{X}/Y)(-1)\oplus h(Y/Y)\xrightarrow{\beta} h(\tilde{Y}/Y)$$

where $\alpha=(c_{d-1}(E)\circ g^*,-i_*)$, $\quad\beta=j_*+f^*$, and $\gamma=g_*$ satisfies $\gamma\circ\alpha=id$.

Note the pairs $(X,\tilde{X})$, (X,Y), $(\tilde{X},\tilde{Y})$, and $(Y,\tilde{Y})$ are all admissible over Y; in addition the triples $(X,\tilde{X},X)$, (X,Y,X), $(X,\tilde{X},\tilde{Y})$, and $(X,Y,\tilde{Y})$, which are relevant for the compositions $\gamma\circ\alpha$ and $\beta\circ\alpha$, are all admissible. Thus one can apply (3.6) and obtains maps α_*, β_*, and γ_* between the higher Chow groups, with the relation $\gamma_*\alpha_*=id$, $\beta_*\alpha_*=0$.

(3.9) For an alternative proof of Theorem (1.1), we take the self-intersection formula in (3.6.1), with $S=X$. The triple (X,Y,X) is admissible. Thus (3.6.1) implies the corresponding equality for maps between higher Chow groups. (Recall Theorem (1.2), (b)-(e) follows from (1.1) and (1.2)(a), (f).)

§4. **Left resolutions of relative motives.**

We show the left resolution $X\mapsto L(X)$ can be extended to relative motives. Indeed we will construct a partial functor $L:\mathcal{CHM}(S)\to\mathcal{D}(k)$. First note that the definition of the complex $H(K,X)^\bullet$ and the notion of left resolution can be extended in an obvious way to the case of objects in $\mathcal{CHM}(S)$. For a smooth projective variety T, $t\in\mathbb{Z}$ and $(X/S,r)$ in $\mathcal{CHM}(S)$, define

$$H((T,t),(X/S,r))^\bullet := \mathcal{Z}_{\dim X-r+t}(T\times X,-\bullet)\ .$$

It is no different from $H((T,t),(X,r))^\bullet$ defined in §2. By linearity one has the *function complex* $H((T,t),M)^\bullet$ for $M\in\mathcal{CHM}(S)$. For an object $K\in\mathcal{D}(k)$ define

$$H(K,M)^\bullet = Tot\,(K^m,M)^\bullet$$

and $H(K,M):=H^0H(K,M)^\bullet$.

Define a complex of abelian groups

$$\mathrm{Hom}_S((X/S,r),(Y/S,s))^\bullet=\bigoplus_j \mathcal{Z}_{\dim Y_j-s+r}(X\times_S Y_j,-\bullet)\ ,$$

and extend it by linearity to define $\mathrm{Hom}_S(M,N)^\bullet$ for M,N in $\mathcal{CHM}(S)$. Note that

$$H^0\,\mathrm{Hom}_S(M,N)^\bullet=\mathrm{Hom}_{\mathcal{CHM}(S)}(M,N)\ .$$

Recall a pair (X,Y) is said to be admissible if $X\times_S Y$ is smooth. The notion of admissibility obviously extends to pairs of objects in $\mathcal{CHM}(S)$. Similarly for three or more varieties in *(Smooth Q-Proj*; *proj*/S) (or objects of $\mathcal{CHM}(S)$) we have the notion of admissibility.

Assume now (M,N,L) is an admissible triple in $\mathcal{CHM}(S)$. There is a partially defined map (defined on a quasi-isomorphic subcomplex)

$$\mathrm{Hom}_S(M,N)^\bullet \otimes \mathrm{Hom}_S(N,L)^\bullet \dashrightarrow \mathrm{Hom}_S(M,L)^\bullet\ .$$

On 0-th cohomology this induces the composition map in $\mathcal{CHM}(S)$.

There are compositions with $H(K,M)^\bullet$ from right and left. One has partially defined maps

$$\mathrm{Hom}(K',K)^\bullet \otimes H(K,M)^\bullet \dashrightarrow H(K',M)^\bullet, \qquad v\otimes\alpha \mapsto \alpha\circ v,$$

$$H(K,M)^\bullet \otimes \mathrm{Hom}_S(M,M')^\bullet \dashrightarrow H(K,M')^\bullet, \qquad \alpha\otimes u \mapsto u\circ\alpha\ .$$

Both are defined on quasi-isomorphic subcomplexes. We have associativity as follows at chain level and on 0-th cohomology:

$$(u\circ\alpha)\circ v = u\circ(\alpha\circ v)\ ,$$

$$(u\circ u')\circ\alpha = u\circ(u'\circ\alpha), \quad \alpha\circ(v\circ v') = (\alpha\circ v)\circ v'\ .$$

(4.1) **Definition.** Let $M\in \mathrm{Ob}\,\mathcal{CHM}(S)$. A pair (L,α) where $L\in\mathcal{D}(k)$, $\alpha\in H(L,M)$ is a *left resolution* of M if for any $K\in\mathcal{D}(k)$ the map

$$\alpha\circ(-) : \mathrm{Hom}_{\mathcal{D}(k)}(K,L)\to H(K,M)$$

is an isomorphism. A left resolution is unique up to unique isomorphism.

The proof of the following theorem is parallel to that of (2.2).

(4.2) **Theorem.** *(1) Each object M of $\mathcal{CHM}(S)$ has a left resolution $L(M)$. If (M,N) is an admissible pair of objects of $\mathcal{CHM}(S)$ and $u\in\mathrm{Hom}_{\mathcal{CHM}(S)}(M,N)$, there exists a unique morphism $L(u): L(M)\to L(N)$ such that the following square commutes.*

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}(k)}(K,L(M)) & \longrightarrow & H(K,M) \\ {\scriptstyle L(u)}\downarrow & & \downarrow{\scriptstyle u\circ(-)} \\ \mathrm{Hom}_{\mathcal{D}(k)}(K,L(N)) & \longrightarrow & H(K,N) \end{array}\ .$$

If (M,N,L) is an admissible triple, $u\in\mathrm{Hom}(M,N)$ and $v\in\mathrm{Hom}(N,L)$, then $L(v\circ u) = L(v)L(u)$.

(2) For $X\in$ (Smooth Q-Proj; proj/S*), one has*

$$L((X/S,r)) = L((X,r)) = h(X)(r)[2r]$$

where $L((X,r))$ is the left resolution of (X,r) defined in §2.

(3) If $f: X\to Y$ is a map of objects in (Smooth Q-Proj; proj/S*), and $f^*: h(Y/S)\to h(X/S)$ the corresponding morphism in $\mathcal{CHM}(S)$, then the induced morphism $L(f^*): h(Y)\to h(X)$ coincides with f^* in §2. For a proper map $f: X\to Y$ of such varieties over S, and the*

morphism $f_*: h(X/S)(\dim X) \to h(Y/S)(\dim Y)$*, the induced* $L(f_*): h(X)(\dim X)[2\dim X] \to h(Y)(\dim Y)[2\dim Y]$ *coincides with the* f_* *in §2.*

For $\alpha \in \mathrm{CH}^r(X)$ *and the corresponding morphism* $C_\alpha : h(X/S) \to h(X/S)(r)$*, the induced morphism* $L(C_\alpha): h(X) \to h(X)(r)[2r]$ *coincides with the* C_α *in §2.*

§5. **The blow-up formula for mixed motives.**

We give analogues of the results in §2 in the category $\mathcal{D}(k)$.

(5.1) **Theorem.** *Let* X *be a smooth quasi-projective variety,* E *a locally free sheaf of rank* $r+1$ *on* X*, and* $\mathbb{P}(E)$ *the associated projective bundle. Then there is a canonical isomorphism in* $\mathcal{D}(k)$

$$h(\mathbb{P}(E)) = h(X) \oplus h(X)(-1)[-2] \oplus \cdots \oplus h(X)(-r)[-2r] \ .$$

Proof. Consider the projective bundle formula (3.3) for the projective bundle $P = \mathbb{P}(E)$ over X, viewed as relative motives over X:

$$h(\mathbb{P}(E)/X) = h(X/X) \oplus h(X/X)(-1) \oplus \cdots \oplus h(X/X)(-r) \ .$$

Recall the isomorphism is given by φ and ψ. Apply the left resolution L of §4 to the two sides. Since the pairs $(\mathbb{P}(E), X)$ and $(X, \mathbb{P}(E))$ are admissible over X, one has the induced morphisms $L(\varphi)$, $L(\psi)$ between $h(\mathbb{P}(E))$ and $h(X) \oplus h(X)(-1)[-2] \oplus \cdots \oplus h(X)(-r)[-2r]$. The triples $(\mathbb{P}(E), X, \mathbb{P}(E)\,)$ and $(X, \mathbb{P}(E), X)$ are admissible, so the identities $\psi \circ \varphi = id$, $\varphi \circ \psi = id$ imply the corresponding identities for $L(\varphi)$ and $L(\psi)$. □

(5.2) **Theorem.** *Let* $i: Y \to X$ *be as in (1.1). Then we have* $i^* \circ i_* = C(c_d(N)) : h(Y) \to h(Y)(d)[2d]$ *.*

(5.3) **Theorem.** *Under the same assumptions as in (1.2), there is a split exact sequence in* $\mathcal{D}(k)$

$$h(X)(-d)[-2d] \xrightarrow{\alpha} h(\tilde{X})(-1)[-2] \oplus h(Y) \xrightarrow{\beta} h(\tilde{Y})$$

where $\alpha = (c_{d-1}(E) \circ g^*, -i_*)$, $\quad \beta = j_* + f^*$*. A left inverse of* α *is given by* $\gamma = g_*$*.*

(5.4) **Corollary.** *There is a canonical isomorphism in* $\mathcal{D}(k)$

$$h(\tilde{Y}) = h(Y) \oplus \bigoplus_{i=1}^{d-1} h(X)(-i)[-2i] \ .$$

(5.5) **Remark.** In [Ma] analogous results are proved for smooth projective varieties and their motives (they are objects in the category of Chow motives – motives with respect to Chow groups). The category of Chow motives $\mathit{CH}\mathcal{M}(k)$ is a full subcategory of $\mathcal{D}(k)$, and the natural functor h : *(Smooth Proj* $/k)^{opp} \to \mathit{CH}\mathcal{M}(k)$ that exists by construction is compatible with h : *(Smooth Proj* $/k)^{opp} \to \mathcal{D}(k)$. The Lefschetz object $\mathbb{L}$ in [Ma] is $\mathbb{Z}(-1)[-2]$ in $\mathcal{D}(k)$. Thus the above results are compatible with those in [Ma].

See also [FV] and [Le] for the blow-up sequences similar to (5.3).

There is an alternative proof of (5.2) and (5.3), which is obtained by modifying the proof in [SGA] as follows: change the Chow groups to motives, and maps $f^*, f_*, \alpha \cdot (-)$ between Chow groups by morphisms $f^*, f_*, C(\alpha)$ between motives. We include only the proof of the self-intersection formula.

To avoid confusion, in the rest of this section $\mathbb{P}(E)$ denotes $\operatorname{Proj}\operatorname{Sym}(\mathcal{E})$, as in [SGA]. But we keep writing N for $N_X Y$.

(5.6)**Proposition.** *Let S be a smooth quasi-projective variety. For E a locally free sheaf of rank $r+1$, one has $p : P = \mathbb{P}(E) \to S$ the associated projective bundle, and the canonical invertible sheaf $\mathcal{O}_P(1)$. Let F be the locally free sheaf of rank r determined by the exact sequence*

$$0 \to F \to p^*E \to \mathcal{O}_P(1) \to 0 \ .$$

Write $\xi = c_1(\mathcal{O}_P(1)) \in \mathrm{CH}^1(P)$.

(1) One has

$$p_* \circ c_r(F^\vee) \circ p^* = id : h(S) \to h(S) \ . \tag{5.6.1}$$

(2) Assume $E = N \oplus \mathcal{O}_S$, where N is locally free of rank r. Then $\mathbb{P}(E)$ is a compactification of the vector bundle $\mathbb{V}(N)$. Let $s : S \to \mathbb{P}(E)$ be the zero section. One has:

$$s_* = c_r(F^\vee) \circ p^* = C(\sum_{0 \le i \le r} p^*(c_i(N^\vee)) \cdot \xi^{r-i}) \circ p^* : h(S) \to h(P)((r)) \ , \tag{5.6.2}$$

$$s^* \circ s_* = c_r(N^\vee) : h(S) \to h(S)((r)) \ , \tag{5.6.3}$$

$$s^* = p_* \circ c_r(F^\vee) : h(P) \to h(S) \ . \tag{5.6.4}$$

Proof. (1) One has $p_*(c_r(F^\vee)) = 1$, the fundamental class of S, [Fu, Example (3.3.1)]. Use (2.3.2).

(2) We have

$$\begin{aligned} s_* &= s_* \circ s^* \circ p^* \\ &= C(s_*(1)) \circ p^* \qquad \text{by (2.3.2).} \end{aligned}$$

One has the following identity; the first equality holds by [Gro, Lemma 3], and the second equality by the Whitney formula.

$$s_*(1) = \sum_{i=o}^{r} p^*(c_i(N^\vee)) \cdot \xi^{r-i} = c_r(F^\vee) \ .$$

Next composing (5.6.2) with s^* gives

$$s^* \circ s_* = s^* \circ C(\sum_{i=0}^{r} p^*(c_i(N^\vee)) \cdot \xi^{r-i}) \circ p^* \ .$$

In view of (2.3.4), we have only to show $s^*(\xi) = 0$. This follows from $\mathcal{O}_P(1)|_{\mathbb{V}(N)} = \mathcal{O}_P$.

Now we show (5.6.4). It suffices to show the morphisms s^* and $p \circ c_r(F^\vee)$ are equal after composing with $\xi^i \circ p^* : h(S)((-i)) \to h(P)$, for $i = 0, \cdots, r$. On the one hand, using (2.3.4)

$$s^* \circ \xi^i \circ p^* = C(s^*(\xi^i)) \circ s^* \circ p^* = C(s^* \xi^i) = \begin{cases} 0 & i > 0, \\ 1 & i = 0. \end{cases}$$

On the other hand, by (2.3.2) and $\xi \cdot c_r(F^\vee) = c_{r+1}(p^*(N^\vee) \bigoplus \mathcal{O}) = 0$,

$$p_* \circ c_r(F^\vee) \circ \xi^i \circ p^* = C(p_*(c_r(F^\vee) \cdot \xi^i)\,) = \begin{cases} 0 & i > 0, \\ 1 & i = 0. \end{cases}$$

□

For an alternative proof of the self-intersection formula (3.6.1) we need some preliminary constructions. Let $u : X \to Z = X \times \mathbb{P}^1$ be the closed immersion $t \mapsto (t, 0)$, and $i_1 := u \circ i : Y \xrightarrow{i} X \xrightarrow{u} Z$. Let $\hat{N} = \mathbb{P}(N^\vee \bigoplus \mathcal{O}_Y)$, and $f_1 : Z' \to Z$ be the blow-up along Y; one has a Cartesian diagram

$$\begin{array}{ccc} \hat{N} & \xrightarrow{j_1} & Z' \\ {\scriptstyle g_1}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{i_1} & Z\ . \end{array}$$

One has an immersion $\alpha = i \times id : W = Y \times \mathbb{P}^1 \to X \times \mathbb{P}^1$. Let W' be the strict transform of W under f_1; the map f_1 restricts to an isomorphism $f_2 : W' \to W$, and the intersection $\hat{N} \cap W'$ is isomorphic to Y. Let $\beta : W' \to Z'$ be the immersion, $\gamma : Y \to Y \times \mathbb{P}^1$ the restriction of u, and $\delta : Y \to W'$ the induced immersion.

$$\begin{array}{ccccc} Y & \xrightarrow{\delta} & W' & \xrightarrow{\beta} & Z' \\ {\scriptstyle id}\downarrow & & {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{\gamma} & Y \times \mathbb{P}^1 & \xrightarrow{\alpha} & Z \end{array}$$

Let X' be the strict transform of X, and $f : X' \to X$ be the induced morphism. The intersection $\hat{N} \cap X'$ is $Y' = \mathbb{P}(N^\vee)$. One has a commutative diagram

$$\begin{array}{ccccc} Y' & \xrightarrow{j} & X' & \xrightarrow{v} & Z' \\ {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y & \xrightarrow{i} & X & \xrightarrow{u} & Z\ . \end{array}$$

Label the map as indicated; also let $k : Y' \to \hat{N}$ be the immersion. Let E be defined by the exact sequence

$$0 \to E \to g_1^*(N^\vee \bigoplus \mathcal{O}_Y) \to \mathcal{O}_{\hat{N}}(1) \to 0$$

and $\bar{\xi} := c_1(\mathcal{O}_{\hat{N}}(1))$. The following is easy to verify (see [SGA, (9.3)]).

(5.7)**Proposition.** *We have the following identities in Chow groups:*

$$j_1^* \beta_*(1) = c_d(E^\vee)\ , \tag{5.7.1}$$

$$j_1^* v_*(1) = \bar{\xi}\ , \tag{5.7.2}$$

and

$$f_1^* u_*(1) = j_{1*}(1) + v_*(1). \tag{5.7.1}$$

(5.8)**Proposition.** *The composition*

$$c_d(E^\vee) \circ j_1^* \circ j_{1*} : h(\hat{N}) \to h(\hat{N})((d+1))$$

is zero.

PROOF. We show the composition of the map with the isomorphism $\bigoplus_{i=0}^{d} h(Y)((-i)) \to h(\hat{N})$ is zero. Namely we are to show the compositions

$$h(Y)((-i)) \xrightarrow{\bar{\xi}^i \circ g_1^*} h(\hat{N}) \xrightarrow{c_d(E^\vee)\circ j_1^* \circ j_{1*}} h(\hat{N})((d+1))$$

are zero. For $i > 0$, they are zero by the following calculation and $\xi \cdot c_d(E^\vee) = 0$.

$$\begin{array}{lll} j_1^* \circ j_{1*} \circ \bar{\xi}^i \circ g_1^* & = j_1^* \circ j_{1*} \circ C\left((j_1^* v_*(1))^i \right) \circ g_1^* & \text{by} \quad \bar{\xi} = j_1^* v_*(1) \\ & = j_1^* \circ C(v_*(1)^i) \circ j_{1*} \circ g_1^* & \text{by} \quad (2.3.4) \\ & = C\left(j_1^*(v_*(1)^i) \right) \circ j_1^* \circ j_{1*} \circ g_1^* & \text{by} \quad (2.3.3) \\ & = \bar{\xi}^i \circ j_1^* \circ j_{1*} \circ g_1^* \, . & \end{array}$$

Consider now the case $i = 0$. Let $b_i : h(Y) \to h(Y)((-i+1))$ be morphisms such that the following square commutes:

$$\begin{array}{ccc} h(Y) & \xrightarrow{j_1^* \circ j_{1*} \circ g_1^*} & h(\hat{N})((1)) \\ {\scriptstyle id}\downarrow & & \downarrow \cong \\ h(Y) & \xrightarrow{(b_0, \cdots, b_d)} & \bigoplus_{i=0}^{d} h(Y)((-i+1)) \, . \end{array}$$

One must show $b_0 : h(Y) \to h(Y)((1))$ is zero. Write

$$j_1^* \circ j_{1*} \circ g_1^* = g_1^* \circ b_0 + C(\bar{\xi}) \circ z$$

with a morphism $z : h(Y) \to h(\hat{N})$. Applying $j_{1*} \circ$ to both sides yields

$$C(j_{1*}(1)) \circ j_{1*} \circ g_1^* = j_{1*} \circ g_1^* \circ b_0 + C(v_*(1)) \circ j_{1*} \circ z \, .$$

Substituting $f_1^* u_*(1) = j_{1*}(1) + v_*(1)$ one has

$$j_{1*} \circ g_1^* \circ b_0 = C(f_1^* u_*(1)) \circ j_{1*} \circ g_1^* - C(v_*(1)) \circ [j_{1*} \circ g_1^* + j_{1*} \circ z] \, .$$

We have $C(f_1^* u_*(1)) \circ j_{1*} = j_{1*} \circ C(j_1^* f_1^* u_*(1))$ by (2.3.3), and $j_1^* f_1^* u_*(1) = 0$, as can be seen by "moving" a cycle from $X \times \{0\}$ to $X \times \{1\}$. Thus

$$j_{1*} \circ g_1^* \circ b_0 = -C(v_*(1)) \circ [j_{1*} \circ g_1^* + j_{1*} \circ z] \, .$$

By $\beta^* v_*(1) = 0$, one has $\beta^* \circ C(v_*(1)) = 0$, so $\beta^* \circ j_{1*} \circ g_1^* \circ b_0 = 0$. By (2.3.5), $\beta^* \circ j_{1*} = \delta_* \circ s^* = 0$, hence

$$\delta_* \circ s^* \circ g_1^* \circ b_0 = \delta_* \circ b_0 = 0$$

thus $\gamma_* \circ b_0 = 0$. Since $\gamma_* : h(Y) \to h(Y \times \mathbb{P}^1)((1))$ is a split monomorphism, $b_0 = 0$. □

(5.9) We now show the self-intersection formula. Let $p : X \times \mathbb{P}^1 \to X$ and $p' : W' \to Y$ be the projections. We claim there is a morphism $w : h(Y) \to h(\hat{N})((d-1))$ such that

$$f_1^* \circ p^* \circ i_* = \beta_* \circ p'^* + j_{1*} \circ w : h(Y) \to h(Z')((d)) \, .$$

Indeed if $m : Z' - \bar{N} \to Z'$ is the open immersion, one has $m^* \circ (f_1^* \circ p^* \circ i_* - \beta_* \circ p'^*) = 0$, so the claim follows from the localization sequence. We show $i^* \circ i_* = s^* \circ s_* : h(Y) \to h(Y)((d))$. Indeed

$$\begin{array}{lll} i^* \circ i_* & = g_{1*} \circ c_d(E^\vee) \circ g_1^* \circ i^* \circ i_* & \text{by} \quad (5.6.1) \\ & = g_{1*} \circ c_d(E^\vee) \circ j_1^* \circ f_1^* \circ p^* \circ i_* & \\ & = g_{1*} \circ c_d(E^\vee) \circ j_1^* \circ (\beta_* \circ p'^* + j_{1*} \circ g_1^*) & \\ & = g_{1*} \circ c_d(E^\vee) \circ j_1^* \circ \beta_* \circ p'^* & \text{by} \quad (5.8) \\ & = s^* \circ j_1^* \circ \beta_* \circ p'^* & \text{by} \quad (5.6.4) \\ & = s^* \circ s_* \, . & \end{array}$$

The last equality follows from $j_1^* \circ \beta_* \circ p'^* = s_* \circ \delta^* \circ p'^* = s_*$ (note $j_1^* \circ \beta_* = s_* \circ \delta^*$ by (2.3.5)). We conclude by (5.6.3).

References.

[Bl 1] S. Bloch, Algebraic cycles and higher K-theory, Adv. in Math. 61 (1986), 267–304.

[Bl 2] S. Bloch, The moving lemma for higher Chow groups, J. Algebraic Geom. 3 (1994), 537–568.

[Bl 3] S. Bloch, Some notes on elementary properties of higher chow groups, including functoriality properties and cubical chow groups, preprint on Bloch's home page.

[CH] A. Corti and M. Hanamura, Motivic decomposition and intersection Chow groups I, Duke Math. J. 103 (2000), 459–522.

[FV] E. Friedlander and V. Voevodsky, Bivariant cycle cohomology, Cycles, transfers, and motivic homology theories (by V. Voevodsky, A. Suslin, and E. M. Friedlander), Princeton Univ. Press, Princeton, NJ, 2000.

[Fu] W. Fulton, Intersection Theory, Springer-Verlag, Berlin 1984.

[Gro] A. Grothendieck, La théorie de classes de Chern, Bull. Soc. Math. France 86 (1958), 137–154.

[Ha 1] M. Hanamura, Mixed motives and algebraic cycles I, Math. Res. Lett. 2 (1995), 811–821; II, Invent. Math. 158 (2004), 105–179; III, Math. Res. Lett. 6 (1999), 61–82.

[Ha 2] M. Hanamura, Homological and cohomological motives of algebraic varieties, Invent. Math. 142(2000), 319–349.

[LMS] A.T. Lascu, D. Mumford and D.B. Scott, The self-intersection formula and the 'formule-clef', Math. Proc. Cambridge Philos. Soc. 78(1975), 117–123.

[Le] M. Levine, Mixed Motives, Mathematical Surveys and Monographs, vol. 57, 1998, American Mathematical Society, RI, 1998.

[Ma] Y.I. Manin, Correspondences, motifs and monoidal transformations, Math. USSR Sbornik 6 (1968), 439–470.

[SGA] Exposé VII in, Séminaire de Géométrie Algébrique du Bois-Marie 1965-66 (SGA 5), Cohomologie ℓ-adique et Fonctions L, Lecture Notes in Math. 589, Springer-Verlag, Berlin-New York, 1977.

Department of Mathematics
Tohoku University
Aramaki Aoba-ku, 980-8587, Sendai, Japan