Relative algebraic correspondences
and mixed motivic sheaves

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Abstract
We introduce the notion of a quasi DG category, and give a procedure to construct a triangulated category associated to it. Then we apply it to the construction of the triangulated category of mixed motivic sheaves over a base variety.

Introduction. We will introduce the notion of a quasi DG category, generalizing that of a DG category. To a quasi DG category satisfying certain additional conditions, we associate another quasi DG category, the quasi DG category of $C$-diagrams. We then show the homotopy category of the quasi DG category of $C$-diagrams has the structure of a triangulated category (see §1).

The main example of a quasi DG category comes from algebraic geometry, as explained in §2. We establish a theory of complexes of relative correspondences; it generalizes the theory of complexes of correspondences of smooth projective varieties, as developed in [6]. The class of smooth quasi-projective varieties equipped with projective maps to a fixed quasi-projective variety $S$, and the complexes of relative correspondences between them constitute a quasi DG category, denoted $\text{Symb}(S)$.

We apply the above procedure to $\text{Symb}(S)$ to obtain $\mathcal{D}(S)$, the triangulated category of mixed motives over $S$. If the base variety is the Spec of the ground field, this coincides with the triangulated category of motives as in [6].

The full details of this article will appear elsewhere (see [8] for §2, [9] for §1).

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Notation and conventions.  (a) A double complex $A = (A^{ij}; d^i, d^j)$ is a bi-graded abelian group with differentials $d^i$ of degree $(1, 0)$, $d^j$ of degree $(0, 1)$, satisfying $d^i d^j + d^j d^i = 0$. Its total complex $\text{Tot}(A)$ is the complex with $\text{Tot}(A)^k = \bigoplus_{i+j=k} A^{ij}$ and the differential $d = d^i + d^j$.

Let $(A, d_A)$ and $(B, d_B)$ be complexes. Then the tensor product complex $A \otimes B$ is the graded abelian group with $(A \otimes B)^n = \oplus_{i+j=n} A^i \otimes B^j$, and with differential $d$ given by

$$d(x \otimes y) = (-1)^{deg_y} dx \otimes y + x \otimes dy.$$  

Note this differs from the usual convention. Alternatively one obtains the same complex by viewing $A \otimes B$ as a double complex with differentials $(-1)^2 d \otimes 1$ and $1 \otimes d$ and taking its total complex.

More generally for $n \geq 2$ one has the notion of an $n$-tuple complex. An $n$-tuple complex is a $\mathbb{Z}^n$-graded abelian group $A^{i_1, \ldots, i_n}$ with differentials $d_1, \ldots, d_n$, $d_k$ raising $i_k$ by 1, such that for $k \neq \ell$, $d_k d_\ell + d_\ell d_k = 0$. A single complex $\text{Tot}(A)$, called the total complex, is defined. For $n$ complexes $A^*_1, \ldots, A^*_n$, the tensor product $A^*_1 \otimes \cdots \otimes A^*_n$ is an $n$-tuple complex; one can take its total complex as well.

(b) Let $I$ be a non-empty finite totally ordered set (we will simply say a finite ordered set), so $I = \{i_1, \ldots, i_n\}, i_1 < \cdots < i_n$, where $n = |I|$. Let $\text{in}(I) = i_1$, $\text{tm}(I) = i_n$, and $\tilde{I} = I - \{\text{in}(I), \text{tm}(I)\}$. For example, for a positive integer $n$, $I = [1, n] = \{1, \ldots, n\}$ is finite ordered set. In this case, if $n \geq 2$, $\tilde{I} = (1, n) := \{2, \ldots, n-1\}$. If $I = \{i_1, \ldots, i_n\}$, a subset $I'$ of the form $[i_a, i_b] = \{i_a, \ldots, i_b\}$ is called a sub-interval.

Given a subset of $\tilde{I}$, $\Sigma = \{i_1, \ldots, i_{a-1}\}$, where $i_1 < i_2 < \cdots < i_{a-1}$, one has a decomposition of $I$ into the sub-intervals $I_1, \ldots, I_a$, where $I_k = [i_{k-1}, i_k]$, with $i_0 = i_1, i_a = i_n$. Thus the sub-intervals satisfy $I_k \cap I_{k+1} = \{i_k\}$ for $k = 1, \ldots, a-1$. The sequence $I_1, \ldots, I_a$ is called the segmentation of $I$ corresponding to $\Sigma$.

§1. Quasi DG categories and triangulated categories.

The notion of a quasi DG category is a generalization of that of a DG category. Recall that a DG category is an additive category $\mathcal{C}$, such that for a pair of objects $X, Y$ the group of homomorphisms $F(X, Y)$ has the structure of a complex, and the composition $F(X, Y) \otimes F(Y, Z) \to F(X, Z)$ is a map of complexes.
(1.1) **Definition.** A quasi DG category $\mathcal{C}$ consists of data (i)-(iii), satisfying the conditions (1)-(5). When necessary we will also impose additional structure (iv),(v), satisfying (6)-(11).

(i) **The class of objects $\text{Ob}(\mathcal{C})$**. There is a distinguished object $O$, called the zero object. There is direct sum of objects $X \oplus Y$, and one has $(X \oplus Y) \oplus Z = X \oplus (Y \oplus Z)$.

(ii) **Multiple complexes $F(X_1, \ldots, X_n)$**. For each sequence of objects $X_1, \ldots X_n$ ($n \geq 2$), a complex of free abelian groups $F(X_1, \ldots, X_n)$.

For a subset $S \subseteq (1, n)$, let $I_1, \ldots, I_n$ be the segmentation of $I = [1, n]$ corresponding to $S$, and $F(X_1, \ldots, X_n|I) := F(I_1) \otimes \cdots \otimes F(I_n)$; this is an $\alpha$-tuple complex. More generally, for a finite ordered set $I$ with cardinality $\geq 2$ and a sequence of objects $(X_i)_{i \in I}$, one has $F(I) = F(I; X)$ and $F(I \upharpoonright S) = F(I \upharpoonright S; X)$.

(iii) **Multiple complexes $F(X_1, \ldots, X_n|S)$ and maps $\iota_S$, $\sigma_{SS'}$ and $\varphi_K$**.

(1) We require given a quasi-isomorphic multiple subcomplex of free abelian groups

$$
\iota_S : F(X_1, \ldots, X_n|S) \hookrightarrow F(X_1, \ldots, X_n| \upharpoonright S).
$$

We assume $F(X_1, \ldots, X_n|\emptyset) = F(X_1, \ldots, X_n)$. The $F(X_1, \ldots, X_n|S)$ is additive in each variable, namely the following properties are satisfied: If a variable $X_i = O$, then it is zero. If $X_1 = Y_1 \oplus Z_1$, then one has a direct sum decomposition of complexes

$$
F(Y_1 \oplus Z_1, X_2, \ldots, X_n|S) = F(Y_1, \ldots, X_n|S) \oplus F(Z_1, \ldots, X_n|S).
$$

The same for $X_n$. If $1 < i < n$ and $X_i = Y_i \oplus Z_i$, then there is a direct sum decomposition of complexes

$$
\begin{align*}
F(X_1, \ldots, X_{i-1}, Y_i \oplus Z_i, X_{i+1}, \ldots, X_n|S) & = F(X_1, \ldots, Y_i, \ldots, X_n|S) \\
\oplus F(X_1, \ldots, Z_i, \ldots, X_n|S) & \oplus F(X_1, \ldots, Y_i|S_1) \otimes F(Z_i, \ldots, X_n|S_2) \\
\oplus F(X_1, \ldots, Z_i|S_1) \otimes F(Y_i, \ldots, X_n|S_2)
\end{align*}
$$

where $S_1, S_2$ is the partition of $S$ by $i$, namely $S_1 = S \cap (1, i)$, $S_2 = S \cap (i, n)$. We often refer to the last two terms as the *cross terms*. (Note the complex
$F(X_1, \cdots, X_n|S)$ is additive in this sense.) The inclusion $\iota_S$ is compatible with the additivity.

For a subset $T \subset S$, if $I_1, \cdots, I_c$ is the segmentation corresponding to $T$, and $S_i = S \cap I_i$, one requires there is an inclusion of multiple complexes

$$F(I|S) \subset F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c)$$

(1.1.1)

where the latter group is viewed as a subcomplex of $F(I|S)$ by the tensor product of the inclusions $\iota_{S_i} : F(I_i|S_i) \hookrightarrow F(I_i|S_i)$.

(2) For $S \subset S'$ given a surjective quasi-isomorphism of multiple complexes

$$\sigma_{SS'} : F(X_1, \cdots, X_n|S) \to F(X_1, \cdots, X_n|S') .$$

For $S \subset S' \subset S''$, $\sigma_{SS''} = \sigma_{SS'} \circ \sigma_{S'S''}$. The $\sigma_{SS'}(X_1, \cdots, X_n)$ is additive in each variable, namely if $X_i = Y_i \oplus Z_i$, then $\sigma_{SS'}(X_1, \cdots, X_n)$ is the direct sum of the maps $\sigma_{SS'}(X_1, \cdots, Y_i, \cdots, X_n)$, $\sigma_{SS'}(X_1, \cdots, Z_i, \cdots, X_n)$, and the maps

$$\begin{align*}
\sigma_{S_1S'_1} \otimes \sigma_{S_2S'_2} : & F(X_1, \cdots, Y_i|S_1) \otimes F(Z_i, \cdots, X_n|S_2) \\
& \to F(X_1, \cdots, Y_i|S'_1) \otimes F(Z_i, \cdots, X_n|S'_2), \\
\sigma_{S_1S'_1} \otimes \sigma_{S_2S'_2} : & F(X_1, \cdots, Z_i|S_1) \otimes F(Y_i, \cdots, X_n|S_2) \\
& \to F(X_1, \cdots, Z_i|S'_1) \otimes F(Y_i, \cdots, X_n|S'_2),
\end{align*}$$

on the cross terms.

The $\sigma$ is assumed compatible with the inclusion in (1.1.1): If $S \subset S'$ and $S'_i = S' \cap I_i$ the following commutes:

$$\begin{array}{c}
F(I|S) \hookrightarrow F(I_1|S_1) \otimes \cdots \otimes F(I_1|S_1) \\
\sigma_{SS'} \downarrow \otimes \sigma_{S_1S'_1} \\
F(I|S') \hookrightarrow F(I_1|S'_1) \otimes \cdots \otimes F(I_1|S'_1). \\
\end{array}$$

We write $\sigma_S = \sigma_{\emptyset S} : F(I) \to F(I|S)$. The composition of $\sigma_S$ and $\iota_S$ is denoted $\tau_S : F(I) \to F(I|S)$.

(3) For $K = \{k_1, \cdots, k_b\} \subset (1, n)$ disjoint from $S$, a map of multiple complexes

$$\varphi_K : F(X_1, \cdots, X_n|S)$$

$$\to F(X_1, \cdots, \widehat{X_{k_1}}, \cdots, \widehat{X_{k_b}}, \cdots, X_n|S) .$$
If \( K = K' \amalg K'' \) then \( \varphi_K = \varphi_K' \varphi_K'' : F(I|S) \to F(I - K|S) \). The \( \varphi_K \) is additive in each variable: If \( X_i = Y_i \oplus Z_i \), then \( \varphi_K(X_1, \ldots, X_n) \) is the sum of \( \varphi_K(X_1, \ldots, Y_i, \ldots, X_n) \), \( \varphi_K(X_1, \ldots, Z_i, \ldots, X_n) \), and, if \( i \notin K \), the maps

\[
\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \ldots, Y_i \upharpoonright S_1) \otimes F(Z_i, \ldots, X_n \upharpoonright S_2),
\]

\[
\varphi_{K_1} \otimes \varphi_{K_2} \text{ on } F(X_1, \ldots, Z_i \upharpoonright S_1) \otimes F(Y_i, \ldots, X_n \upharpoonright S_2)
\]

on the cross terms (\( S_1, S_2 \) is the partition of \( S \) by \( i \), and \( K_1, K_2 \) is the partition of \( K \) by \( i \)), and if \( i \in K \), the zero maps on the cross terms.

\( \varphi_K \) is assumed to be compatible with the inclusion in \((1.1.1)\): With the same notation as above and \( K_i = K \cap I_i \), the following commutes:

\[
\begin{array}{ccc}
F(I|S) & \to & F(I_1|S_1) \otimes \cdots \otimes F(I_c|S_c) \\
\varphi_K \downarrow & & \downarrow \otimes \varphi_{K_i} \\
F(I - K|S) & \to & F(I_1 - K_1|S_1) \otimes \cdots \otimes F(I_c - K_c|S_c).
\end{array}
\]

If \( K \) and \( S' \) are disjoint the following commutes:

\[
\begin{array}{ccc}
F(I|S) & \xrightarrow{\varphi_K} & F(I - K|S) \\
\sigma_{SS'} \downarrow & & \downarrow \sigma_{SS'} \\
F(I|S') & \xrightarrow{\varphi_K} & F(I - K|S').
\end{array}
\]

(4) (acyclicity of \( \sigma \)) For disjoint subsets \( R, J \) of \( I \) with \(|J| \neq \emptyset\), consider the following sequence of complexes, where the maps are alternating sums of \( \sigma \), and \( S \) varies over subsets of \( J \):

\[
F(I|R) \xrightarrow{\sigma} \bigoplus_{\substack{|S| = 1 \\ S \subseteq J}} F(I|R \cup S) \xrightarrow{\sigma} \bigoplus_{\substack{|S| = 2 \\ S \subseteq J}} F(I|R \cup S) \to \cdots \to F(I|R \cup J) \to 0.
\]

Then the sequence is exact.

(5) (existence of the identity in the ring \( H^0F(X,X) \)) Before stating the condition, note there are composition maps for \( H^0F(X,Y) \) defined as follows. For three objects \( X, Y \) and \( Z \), let

\[
\psi_Y : F(X,Y) \otimes F(Y,Z) \to F(X,Z)
\]
be the map in the derived category defined as the composition $\varphi_Y \circ (\sigma_Y)^{-1}$
where the maps are as in
$$F(X, Y) \otimes F(Y, Z) \xrightarrow{\sigma_Y} F(X, Y, Z) \xrightarrow{\varphi_Y} F(X, Z).$$

The map $\psi_Y$ is verified to be associative, namely the following commutes in
the derived category:
$$F(X, Y) \otimes F(Y, Z) \otimes F(Z, W) \xrightarrow{\psi_Y \otimes \text{id}} F(X, Z) \otimes F(Z, W)$$
$$\xrightarrow{id \otimes \psi_Z} F(X, Y) \otimes F(Y, W) \xrightarrow{\psi_Y} F(X, W).$$

Let $H^0 F(X, Y)$ be the $0$-th cohomology of $F(X, Y)$. $\psi_Y$ induces a map
$$\psi_Y : H^0 F(X, Y) \otimes H^0 F(Y, Z) \rightarrow H^0 F(X, Z),$$
which is associative. If $u \in H^0 F(X, Y)$, $v \in H^0 F(Y, Z)$, we write $u \cdot v$ for
$\psi_Y(u \otimes v)$.

We now require: For each $X$ there is an element $1_X \in H^0 F(X, X)$ such
that $1_X \cdot u = u$ for any $u \in H^0 F(X, Y)$ and $u \cdot 1_X = u$ for $u \in H^0 F(Y, X)$.

(iv) Diagonal elements and diagonal extension.

(6) For each irreducible object $X$ and a constant sequence of objects
$i \mapsto X_i = X$ on a finite ordered set $I$ with $|I| \geq 2$, there is a distinguished
element, called the diagonal element
$$\Delta_X(I) \in F(I) = F(X, \cdots, X)$$
of degree zero and coboundary zero. In particular for $|I| = 2$ we write
$\Delta_X = \Delta_X(I) \in F(X, X)$. One requires:

(6-1) If $S \subset I$, and $I_1, \cdots, I_c$ the corresponding segmentation, one has
$$\tau_S(\Delta_X(I)) = \Delta_X(I_1) \otimes \cdots \otimes \Delta_X(I_c)$$
in $F(I \mid S) = F(I_1) \otimes \cdots \otimes F(I_c)$.

(6-2) For $K \subset I$, $\varphi_K(\Delta_X(I)) = \Delta_X(I - K)$.

(7) Let $I$ be a finite ordered set, $k \in I$, $m \geq 2$, and $I^*$ be the finite
ordered set obtained by replacing $k$ by a finite ordered set with $m$ elements
$\{k_1, \cdots, k_m\}$. If $I = [1, n]$, $I^*$ is $\{1, \cdots, k - 1, k_1, \cdots, k_m, k + 1, \cdots, n\}$.
There is given a map of complexes, called the \textit{diagonal extension},
\[ \text{diag}(I, \Gamma) : F(I) \to F(\Gamma) \]
subject to the following conditions (for simplicity assume \( I = [1, n] \)):

(7-1) If \( k' \neq k \), \( \phi_{k'} \text{diag}(I, \Gamma) = \text{diag}(I - \{k'\}, \Gamma - \{k'\})\phi_k' \), namely the following square commutes:

\[
\begin{array}{ccc}
F(I) & \xrightarrow{\text{diag}(I, \Gamma)} & F(\Gamma) \\
\phi_k' \downarrow & & \phi_k' \downarrow \\
F(I - \{k\}) & \xrightarrow{\text{diag}(I - \{k'\}, \Gamma - \{k'\})} & F(\Gamma - \{k'\}) \\
\end{array}
\]

If \( \ell \in \{k_1, \cdots, k_m\} \), \( \phi_\ell \text{diag}(I, \Gamma) = \text{diag}(I, \Gamma - \{\ell\}) \). If \( m = 2 \) the right side is the identity.

(7-2) If \( k = n \), \( \ell \in \{n_1, \cdots, n_m\} \), let \( I_1', I'' \) be the segmentation of \( \Gamma \) by \( \ell \). Then the following diagram commutes:

\[
\begin{array}{c}
F(I) \\
\text{diag}(I, \Gamma) \downarrow \\
F(I_1') \xrightarrow{\text{diag}(I_1', \Gamma)} F(\Gamma) \\
\end{array}
\]

\[
\begin{array}{c}
\tau_\ell \\
\end{array}
\]

\[
\begin{array}{c}
F(I_1') \xrightarrow{\tau_\ell} F(I_1') \otimes F(I'') \\
\end{array}
\]

The lower horizontal map is \( u \mapsto u \otimes \Delta(I'') \). Note \( I'' \) parametrizes a constant sequence of objects, so one has \( \Delta(I'') \in F(I'') \). Similarly in case \( k = 1 \), \( \ell \in \{1, \cdots, 1_m\} \).

If \( 1 < k < n \) and \( \ell \in \{k_1, \cdots, k_m\} \), let \( I_1, I_2 \) be the segmentation of \( I \) by \( k \), and \( I_1', I_2' \) of \( \Gamma \) by \( \ell \). One then has a commutative diagram:

\[
\begin{array}{ccc}
F(I) & \xrightarrow{\text{diag}(I, \Gamma)} & F(\Gamma) \\
\tau_\ell \downarrow & & \tau_\ell \downarrow \\
F(I_1) \otimes F(I_2) & \xrightarrow{\tau_\ell} & F(I_1') \otimes F(I_2') \\
\end{array}
\]

where the lower horizontal arrow is \( \text{diag}(I_1, I_1') \otimes \text{diag}(I_2, I_2') \).

\textit{Remark.} From (6) and (7) it follows that \( [\Delta_X] \in H^0F(X, X) \) is the identity in the sense of (5). Indeed the following stronger property is satisfied for the maps \( \psi_Y : H^mF(X, Y) \otimes H^nF(Y, Z) \to H^{m+n}F(X, Z) \) for \( m, n \in \mathbb{Z} \), defined in a similar manner as in (5) above.

(5)' For each \( u \in H^nF(X, Y) \), \( n \in \mathbb{Z} \), one has \( 1_X \cdot u = u \). Similarly for \( u \in H^nF(Y, X) \), \( u \cdot 1_X = u \).
(v) The set of generators, notion of proper intersection, and distinguished subcomplexes with respect to constraints.

(8) (the generating set) For a sequence $X$ on $I$, the complex $F(I) = F(I; X)$ is degree-wise $\mathbb{Z}$-free on a given set of generators $S_F(I) = S_F(I; X)$. More precisely $S_F(I) = \bigcap_{p \in \mathbb{Z}} S_F(I)^p$, where $S_F(I)^p$ generates $F(I)^p$. This set is compatible with direct sum in each variable: Assume for an element $k \in I$ one has $X_k = Y_k \oplus Z_k$; let $X'_i = X_i$ for $i \neq k$, and $X''_i = Y_i$ (resp. $X''_i = X_i$ for $i \neq k$, and $X''_i = Z_i$). Then $S_F(I; X) = S_F(I; X') \sqcup S_F(I; X'')$.

(9) (notion of proper intersection.) Let $I$ be a finite ordered set, $I_1, \ldots, I_r$ be almost disjoint sub-intervals of $I$, namely one has $\text{tm}(I_i) \leq \text{in}(I_{i+1})$ for each $i$. Assume given a sequence of objects $X_i$ on $I$. Let $\alpha_i \in S_F(I_i)$ be a set of elements where $i$ varies over a subset $A$ of $\{1, \ldots, r\}$. We are given the condition whether the set $\{\alpha_i\}$ is properly intersecting. The following condition is to be satisfied.

- If $\{\alpha_i | i \in A\}$ is properly intersecting, for any subset $B$ of $A$, $\{\alpha_i | i \in B\}$ is properly intersecting.
- Let $A$ and $A'$ be subsets of $\{1, \ldots, r\}$ such that $\text{tm}(A) < \text{in}(A')$. If $\{\alpha_i | i \in A\}$ and $\{\alpha_i | i \in A'\}$ are both properly intersecting sets, the union $\{\alpha_i | i \in A \cup A'\}$ is also properly intersecting.
- If $\{\alpha_1, \ldots, \alpha_r\}$ is properly intersecting, then for any $i$, writing $\partial \alpha_i = \sum c_{\nu} \beta_{\nu}$ with $\beta_{\nu} \in S_F(I_i)$, each set
  \[ \{\alpha_1, \ldots, \alpha_{i-1}, \beta_{\nu}, \alpha_{i+1}, \ldots, \alpha_r\} \]
  is properly intersecting.
- The condition of proper intersection is compatible with direct sum in each variable. To be precise, under the same assumption as in (8), for a set of elements $\alpha_i \in S_F(I_i; X')$ for $i = 1, \ldots, r$, the set $\{\alpha_i \in S_F(I_i; X')\}_{i}$ is properly intersecting if and only if the set $\{\alpha_i \in S_F(I_i; X)\}_{i}$ is properly intersecting.

Remark. For $I_i$ almost disjoint and elements $\alpha_i \in F(I_i)$, one defines $\{\alpha_i \in F(I_i) | i \in A\}$ to be properly intersecting if the following holds. Write $\alpha_i = \sum c_{\nu} \alpha_{i\nu}$ with $\alpha_{i\nu} \in S_F(I_i)$, then for any choice of $\nu_i$ for $i \in A$, the set $\{\alpha_{i\nu_i} | i \in A\}$ is properly intersecting.
Further, if \( S_i \subset I_i \), one can define the condition of proper intersection for \( \{ \alpha_i \in F(I_i|S_i) | i \in A \} \) by writing each \( \alpha_i \) as a sum of tensors of elements in the generating set.

(10) (description of \( F(I|S) \)) When \( I_1, \ldots, I_r \) is a segmentation of \( I \), namely when \( \text{in}(I_1) = \text{in}(I) \), \( \text{tm}(I_i) = \text{in}(I_{i+1}) \) and \( \text{tm}(I_r) = \text{tm}(I) \), the subcomplex of \( F(I_1) \otimes \cdots \otimes F(I_r) \) generated by \( \alpha_1 \otimes \cdots \otimes \alpha_r \) with \( \{ \alpha_i \} \) properly intersecting is denoted by \( F(I_1) \otimes \cdots \otimes F(I_r) \). If \( S \subset I \) is the subset corresponding to the segmentation, this subcomplex coincides with \( F(I|S) \).

(11)(distinguished subcomplexes) Let \( I \) be a finite ordered set, \( L_1, \ldots, L_r \) be almost disjoint sub-intervals such that \( \bigcup L_i = I \); equivalently, \( \text{in}(L_1) = \text{in}(I) \), \( \text{tm}(L_i) = \text{in}(I_{i+1}) \) or \( \text{tm}(L_i) + 1 = \text{in}(L_{i+1}) \), and \( \text{tm}(L_r) = \text{tm}(I) \). Assume given a sequence of objects \( X, I \), on \( I \). Let \( \text{Dist} \) be the smallest class of subcomplexes of \( F(L_1) \otimes \cdots \otimes F(L_r) \) satisfying the conditions below. It is then required that each subcomplex \( \text{Dist} \) is a quasi-isomorphic subcomplex.

(11-1) A subcomplex obtained as follows is in \( \text{Dist} \). Let \( I_1, \ldots, I_s \) be a set of almost disjoint sub-intervals of \( I \) with union \( I \), that is coarser than \( L_1, \ldots, L_r \); let \( S_i \subset I_i \) such that the segmentations of \( I_i \) by \( S_i \), when combined for all \( i \), give precisely the \( L_i \)’s. Let \( I \hookrightarrow \mathbb{I} \) be an inclusion into a finite ordered set \( \mathbb{I} \) such that the image of each \( I_i \) is a sub-interval. Assume given an extension of \( X \) to \( \mathbb{I} \). Let \( J_1, \ldots, J_s \subset \mathbb{I} \) be sub-intervals of \( \mathbb{I} \) such that the set \( \{ I_i, J_j \} \) is almost disjoint, and \( f_j \in F(J_j|T_j) \), \( j = 1, \ldots, s \) be a properly intersecting set. Then one defines the subcomplex

\[
[F(I_1|S_1) \otimes \cdots \otimes F(I_s|S_s)][i:f],
\]

as the one generated by \( \alpha_1 \otimes \cdots \otimes \alpha_c \), \( \alpha_i \in F(I_i|S_i) \), such that the set \( \{ \alpha_1, \ldots, \alpha_c, f_j \} (j = 1, \ldots, s) \) is properly intersecting. We require it is in \( \text{Dist} \).

The data consisting of \( I \hookrightarrow \mathbb{I}, X \) on \( \mathbb{I} \), \( J_i \subset \mathbb{I} \), and \( f_j \in F(J_j|T_j) \) is called a constraint, and the corresponding subcomplex the distinguished subcomplex for the constraint.

(11-2) Tensor product of subcomplexes in \( \text{Dist} \) is again in \( \text{Dist} \). For this to make sense, note complexes of the form \( F(L_1) \otimes \cdots \otimes F(L_r) \) are closed under tensor products: If \( I' \) is another finite ordered set and \( L'_1, \ldots, L'_s \) are almost disjoint sub-intervals with union \( I' \), then the tensor product

\[
F(L_1) \otimes \cdots \otimes F(I_r) \otimes F(L'_1) \otimes \cdots \otimes F(I'_s)
\]
is associated with the ordered set $I \subset I'$ and almost disjoint sub-intervals $(L_1, \ldots, L_r, L'_1, \ldots, L'_s)$.

(11-3) A finite intersection of subcomplexes in $Dist$ is again in $Dist$.

(1.2) Definition. To a quasi DG category $\mathcal{C}$ one can associate an additive category, called its homotopy category, denoted by $Ho(\mathcal{C})$. Objects of $Ho(\mathcal{C})$ are the same as the objects of $\mathcal{C}$, and $\text{Hom}(X,Y) := H^0F(X,Y)$. Composition of arrows is induced from $\psi_Y$ as in (5) above. The object $O$ is the zero object, and the direct sum $X \oplus Y$ is the direct sum in the categorical sense. $1_X$ gives the identity $X \to X$.

(1.3) Definition. Let $\mathcal{C}$ be a quasi DG category. A $\mathcal{C}$-diagram in $\mathcal{C}$ is an object of the form $K = (K^m; f(m_1, \ldots, m_\mu))$, where $(K^m)$ is a sequence of objects of $\mathcal{C}$ indexed by $m \in \mathbb{Z}$, almost all of which are zero, and

$$f(m_1, \ldots, m_\mu) \in F(K^{m_1}, \ldots, K^{m_\mu}) - \{m_\mu - m_1 - \mu + 1\}$$

is a collection of elements indexed by sequences $(m_1 < m_2 < \cdots < m_\mu)$ with $\mu \geq 2$. We require the following conditions:

(i) For each $j = 2, \ldots, \mu - 1$

$$\tau_{K^m_j}(f(m_1, \ldots, m_\mu)) = f(m_1, \ldots, m_j) \otimes f(m_j, \ldots, m_\mu)$$

in $F(K^{m_1}, \ldots, K^{m_j}) \otimes F(K^{m_j}, \ldots, K^{m_\mu})$.

(ii) For each $(m_1, \ldots, m_\mu)$, one has

$$\partial f(m_1, \ldots, m_\mu) + \sum_t \sum_k (-1)^{m_\mu + \mu + k + t} \varphi_{K^m_k}(f(m_1, \ldots, m_t, k, m_{t+1}, \ldots, m_\mu)) = 0$$

(the sum is over $t$ with $1 \leq t < \mu$, and $k$ with $m_t < k < m_{t+1}$).

For an object $X$ in $\mathcal{C}$ and $n \in \mathbb{Z}$, there is a $C$-diagram $K$ with $K^n = X$, $K^m = 0$ if $m \neq n$, and $f(M) = 0$ for all $M = (m_1, \ldots, m_\mu)$. We write $X[-n]$ for this.

(1.4) Theorem. Let $\mathcal{C}$ be a quasi DG category satisfying the extra conditions (iv),(v) of Definition (1.1). There is a quasi DG category $\mathcal{C}^\Delta$ satisfying the following properties:

(i) The objects are the $C$-diagrams in $\mathcal{C}$.
(ii) For a sequence of $C$-diagrams $K_1, \ldots, K_n$ with $n \geq 2$, as part of the structure of a quasi DG category, one has the corresponding complex of abelian groups $F(K_1, \ldots, K_n)$, and the maps $\iota$, $\sigma$, and $\varphi$. This complex has the following description if $n = 2$ and the diagrams $K_1, K_2$ are “objects of $\mathcal{C}$ with shifts”: For a pair of objects $X, Y$ in $\mathcal{C}$, and $m, n \in \mathbb{Z}$, and the corresponding $C$-diagrams $X[m], Y[n]$, one has a canonical isomorphism of complexes

$$F(X[m], Y[n]) = F(X, Y)[n - m].$$

In particular, in the homotopy category $Ho(\mathcal{C}^\Delta)$ of $\mathcal{C}^\Delta$, one has

$$\text{Hom}_{Ho(\mathcal{C}^\Delta)}(X[m], Y[n]) = H^{n - m}F(X, Y).$$

Further, the map

$$\psi_Y : H^mF(X, Y) \otimes H^nF(Y, Z) \to H^{m+n}F(X, Z)$$

for $m, n \in \mathbb{Z}$, defined using the maps $\sigma$, $\varphi$ and $F(X, Y, Z)$ (see the remark just before (v) in (1.1)) coincides with the map

$$\psi_Y : H^0F(X, Y[m]) \otimes H^0F(Y[m], Z[m + n])$$

$$\to H^0F(X, Z[m + n])$$

defined similarly using the maps $\sigma$, $\varphi$ and $F(X, Y[m], Z[m + n])$, via the isomorphisms $H^mF(X, Y) = H^0F(X, Y[m])$, etc.

(iii) The homotopy category $Ho(\mathcal{C}^\Delta)$ of $\mathcal{C}^\Delta$ has the structure of a triangulated category.

For the proof, we must define the complexes $F(K_1, \ldots, K_n)$ for a sequence of $C$-diagrams, together with maps $\sigma$ and $\varphi$, satisfying the condition (ii) of the theorem, and the axioms (i)-(iii) of a quasi DG category. We then proceed to show that the homotopy category of $\mathcal{C}^\Delta$ is triangulated. If $\mathcal{C}$ is a DG category, there is a procedure to construct a triangulated category, as in [6] and [10]. The present construction may be viewed as its generalization.

§2. The quasi DG category of smooth varieties over a base.

We consider quasi-projective varieties over a field $k$. We refer the reader to [1], [2], [3] for the definition of the cycle complexes and the higher Chow groups of quasi-projective varieties. We will use the integral cubical version,
as in [3]. Thus to a quasi-projective variety \( X \) over \( k \) and \( s \in \mathbb{Z} \), there corresponds the cycle complex \( \mathbb{Z}_s(X, \cdot) \); the group \( \mathbb{Z}_s(X, n) \) is a quotient of the free abelian group of algebraic cycles on \( X \times \Delta^n \) of dimension \( s + n \), meeting faces properly. (See [3] for the precise definition, where the indexing is by codimension.) The variety \( X \) need not be assumed equi-dimensional when we use the indexing by “dimension” instead of codimension. The higher Chow groups are the homology groups of this complex: \( \text{CH}_s(X, n) = H_n \mathbb{Z}_s(X, \cdot) \).

Let \( S \) be a quasi-projective variety. Let \((\text{Smooth}/k, \text{Proj}/S)\) be the category of smooth varieties \( X \) equipped with projective maps to \( S \). A symbol over \( S \) is an object the form

\[
\bigoplus_{a \in A} (X_a/S, r_a)
\]

where \( X_a \) is a collection of objects in \((\text{Smooth}/k, \text{Proj}/S)\) indexed by a finite set \( A \), and \( r_a \in \mathbb{Z} \).

(2.1) **Theorem.** There is a quasi DG category satisfying the conditions (iv), (v), denoted \( \text{Symb}(S) \), with the following properties:

(i) The objects are the symbols over \( S \).

(ii) For a sequence of symbols \( K_1, \ldots, K_n \) with \( n \geq 2 \), as part of the structure of a quasi DG category, one has the corresponding complex of abelian groups \( F(K_1, \ldots, K_n) \), and the maps \( i, \sigma, \text{ and } \varphi \). When the symbols are of the form \( K_i = (X_i/S, r_i) \), the corresponding complex \( F(K_1, \ldots, K_n) \) is quasi-isomorphic to

\[
\mathbb{Z}_{d_i}(X_1 \times_S X_2) \otimes \cdots \otimes \mathbb{Z}_{d_{n-1}}(X_{n-1} \times_S X_n),
\]

with \( d_i = \dim X_{i+1} - r_{i+1} + r_i \), the tensor product of the cycle complexes of the fiber products \( X_i \times_S X_{i+1} \).

We consider \( \text{Symb}(S)^\wedge \), the quasi DG category of \( C \)-diagrams in \( \text{Symb}(S) \), and then take its homotopy category. The resulting category is denoted \( \mathcal{D}(S) \), and called the triangulated category of mixed motives over \( S \). The next theorem follows from (1.3) and (2.1).

(2.2) **Theorem.** For \( X \) in \((\text{Smooth}/k, \text{Proj}/S)\) and \( r \in \mathbb{Z} \), there corresponds an object \( h(X/S)(r) := (X/S, r)[-2r] \) in \( \mathcal{D}(S) \). For two such objects we have

\[
\text{Hom}_\mathcal{D}(S)(h(X/S)(r)[2r], h(Y/S)(s)[2s-n]) = \text{CH}_{\dim Y - s + r} (X \times_SY, n)
\]

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the right hand side being the higher Chow group of the fiber product $X \times_S Y$.

There is a functor

$$h : (\text{Smooth}/k, \text{Proj}/S)^{\text{opp}} \to \mathcal{D}(S)$$

that sends $X$ to $h(X/S)$, and a map $f : X \to Y$ to the class of its graph $[\Gamma_f] \in \text{CH}_{\text{dim}X}(Y \times_S X)$.

References.

[3] — : Some notes on elementary properties of higher chow groups, including functoriality properties and cubical chow groups, preprint on Bloch’s home page.

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